ASYMPTOTICS FOR PRODUCTS OF CHARACTERISTIC POLYNOMIALS IN CLASSICAL β-ENSEMBLES

PATRICK DESROSIERS AND DANG-ZHENG LIU

ABSTRACT. We study the local properties of eigenvalues for the Hermite (Gaussian), Laguerre (Chiral) and Jacobi β-ensembles of $N \times N$ random matrices. More specifically, we calculate scaling limits of the expectation value of products of characteristic polynomials as $N \to \infty$. In the bulk of the spectrum of each β-ensemble, the same scaling limit is found to be $e^{p_1}{}_1F_1$ whose exact expansion in terms of Jack polynomials is well known. The scaling limit at the soft edge of the spectrum for the Hermite and Laguerre β-ensembles is shown to be a multivariate Airy function, which is defined as a generalized Kontsevich integral. As corollaries, when β is even, scaling limits of the k-point correlation functions for the three ensembles are obtained. The asymptotics of the multivariate Airy function for large and small arguments is also given. All the asymptotic results rely on a generalization of Watson's lemma and the steepest descent method for integrals of Selberg type.

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Date: July 25, 2012.
2010 *Mathematics Subject Classification.* 15B52,41A60, 05E05, 33C70.
Key words and phrases. Random Matrices, Beta-ensembles, Jack polynomial, multivariate hypergeometric function, steepest descent method.

1. Introduction

1.1. **β-Ensembles of random matrices.** In this article we consider three classical β-ensembles of Random Matrix Theory, namely the Hermite (Gaussian), Laguerre (Chiral), and Jacobi β-ensembles (HβE, LβE, and JβE for short). Their eigenvalue probability density functions are equal to

$$\frac{1}{G_{\beta,N}}\prod_{1\le i\le N}e^{-\beta x_i^2/2}\prod_{1\le j<k\le N}|x_j-x_k|^\beta,\qquad x_i\in\mathbb{R},\qquad \text{H}\beta\text{E},\tag{1.1}$$

$$\frac{1}{W_{\lambda_1,\beta,N}}\prod_{1\le i\le N}x_i^{\lambda_1}e^{-\beta x_i/2}\prod_{1\le j<k\le N}|x_j-x_k|^\beta,\qquad x_i\in\mathbb{R}_+,\qquad \text{L}\beta\text{E},\tag{1.2}$$

$$\frac{1}{S_N(\lambda_1,\lambda_2,\beta/2)}\prod_{1\le i\le N}x_i^{\lambda_1}(1-x_i)^{\lambda_2}\prod_{1\le j<k\le N}|x_j-x_k|^\beta,\qquad x_i\in(0,1),\qquad \text{J}\beta\text{E}.\tag{1.3}$$

The normalization constants are all special cases of Selberg's celebrated formula [23] and are given in the appendix.

For special values of the Dyson index β, we recover more conventional random matrix ensembles [23, 41]. The $\beta=1,2,4$–ensembles indeed correspond to the ensembles of random matrices whose respective probability measures exhibit orthogonal, unitary or symplectic symmetry.

For general $\beta>0$, Dumitriu and Edelman [13] constructed tri-diagonal real symmetric matrices with independent entries randomly drawn from some specific distributions and whose eigenvalues are distributed according to (1.1) and (1.2). Killip and Nenciu [30] later obtained a similar construction for the JβE. These explicit constructions play a key role in connecting the β-ensembles with one-dimensional stochastic differential equations in the limit $N\to\infty$ [17,45,46]. Many probabilistic quantities of interest such as the global fluctuations, the gap probabilities, and the distribution of the largest eigenvalues were also studied in the limit $N\to\infty$ (see for instance [8,15,25,29,46,50]). More recently, some universality results (see below) concerning the general $\beta>0$ case have been obtained (see [19] and references therein). It was shown for instance that as $N\to\infty$, the eigenvalues in the bulk (middle) of the spectrum of any β-ensemble are correlated, when appropriately rescaled, as the eigenvalues of the Hermite β-ensemble.

Apart from being related to the eigenvalues of random matrices, the densities (1.1)–(1.3) have alternative physical interpretations. Indeed, these densities appeared very recently in theoretical high energy physics [6,9,42,48]. Moreover, densities such as (1.1)–(1.3) are equivalent to the Boltzmann factor for classical log-potential Coulomb gas and to the ground state wave functions squared for Calogero-Sutherland N-body quantum systems of the type A_{N-1}, B_N and BC_N. We refer the reader to Forrester's lectures [22] for more details. The wave functions of the Calogero-Sutherland models are typically written in terms of a very special family of symmetric polynomials, namely the Jack polynomials [27,37,49]. This connection between the β-ensembles and the Jack polynomials has been exploited by many authors and has shown to be very fruitful [4,10–16,18,21,24,32,38–40,43]

1.2. **Products of characteristic polynomials.** Now let X be an $N\times N$ random matrix in some β-ensemble. Our aim is to find exact and explicit expressions for the large N limit of the expectation value of $\prod_{j=1}^n\det(X-s_j)$. We will thus study the following expectation value:

$$K_N(s_1,\dots,s_n)=\Big\langle\prod_{i=1}^N\prod_{j=1}^n(x_i-s_j)\Big\rangle_{x\in\beta\text{E}}\tag{1.4}$$

where $x=(x_1,\dots,x_N)$ denotes the eigenvalues of the random matrix X and where the angle brackets stand for the expectation value. More explicitly, for the densities (1.1)–(1.3),

$$K_N(s_1,\dots,s_n)=$$
$$\frac{1}{Z_N}\int\cdots\int\prod_{i=1}^N\prod_{j=1}^n(x_i-s_j)\exp\Big\{-\frac{\beta}{2}\sum_{j=1}^NV(x_j)\Big\}\prod_{1\le i<j\le N}|x_i-x_j|^\beta\,dx_1\cdots dx_N,\tag{1.5}$$

where Z_N is some normalization constant and

$$V(x_j) = \begin{cases} x_j^2 & \text{H}\beta\text{E} \\ x_j - (2/\beta)\lambda_1 \ln x_j & \text{L}\beta\text{E} \\ -(2/\beta)\lambda_1 \ln x_j - (2/\beta)\lambda_2 \ln(1-x_j) & \text{L}\beta\text{E} \end{cases} \tag{1.6}$$

Actually, we will see that it is more convenient to consider the weighted quantity

$$\varphi_N(s_1,\dots,s_n) = \exp\Big\{-\frac{1}{2}\sum_{j=1}^n V(s_j)\Big\} K_N(s_1,\dots,s_n). \tag{1.7}$$

At this point, it is worth stressing that if β is even and if we let $n = k\beta$, then $\varphi_N(s_1,\dots,s_n)$ gives access to the k-point correlation function [23, 41]:

$$R_{k,N}(x_1,\dots,x_k) = \\ \frac{(k+N)!}{N!}\frac{1}{Z_{k+N}}\int\cdots\int \exp\Big\{-\frac{\beta}{2}\sum_{j=1}^{k+N} V(x_j)\Big\}\prod_{1\le i<j\le k+N}|x_i-x_j|^\beta dx_{k+1}\cdots dx_{k+N} \tag{1.8}$$

Indeed,

$$R_{k,N}(x_1,\dots,x_k) = \frac{(k+N)!}{N!}\frac{Z_N}{Z_{k+N}}\prod_{1\le i<j\le k}(x_i-x_j)^\beta\big[\varphi_N(s_1,\dots,s_n)\big]_{\{s\}\mapsto\{x\}}. \tag{1.9}$$

Here the notation $\{s\}\mapsto\{x\}$ means that the variables s_i are evaluated as follows:

$$s_{(i-1)\beta+j} = x_i \quad \text{for} \quad i=1,\dots,k \quad \text{and} \quad j=1,\dots,\beta. \tag{1.10}$$

For the special values $\beta = 1, 2, 4$, the scaling limits of products of characteristic polynomials are already well established (see [1, 5, 7] and references therein). Thanks to the orthogonal polynomial method, they can be expressed as determinants or Pfaffians of one-variable special functions and their derivatives. The functions in question depend on the bulk or edge of the spectrum we are looking at [20]. Close to the hard edge of the spectrum (which correspond to $s_i = 0$ for Laguerre and $s_i = 0, 1$ for Jacobi ensembles), one gets a Bessel function or equivalently a ${}_0F_1(z)$ function. In the bulk of the spectrum of each ensemble (for instance, about $s_i = 1/2$, $2N$, and 0 for Jacobi, Laguerre, and Hermite, respectively), the formulas involve trigonometric functions or complex functions of exponential type, such as ${}_0F_0(iz) = {}_1F_1(a;a;iz)$. Finally, at the soft edge of the spectrum (i.e., about $s_j = 4N$ for Laguerre and $s_j = \sqrt{2N}$ for Hermite) the scaling limits contain the Airy function Ai. In fact, these three regimes of large N asymptotics (Hard-Bessel, Bulk-Trigonometric, Soft-Airy) correspond to the three most common universality classes for ensembles of random matrices with $\beta = 1, 2, 4$ (e.g., see Chapter 7 in [23]).

Much less is known about the general $\beta > 0$ case. For the HβE, Aomoto [3] and more recently Su [47], obtained the limiting expectation value of the product of $n = 2$ characteristic polynomials respectively in the bulk and at the soft edge of the spectrum. When both n and N are finite but arbitrary, Baker and Forrester [4] proved that $K_N(s_1,\dots,s_n)$ is either a multivariate Jacobi, Laguerre or Hermite polynomials with parameter $\alpha = \beta/2$ depending on whether the density considered is (1.3), (1.2) or (1.1). They also used the theory of multivariate hypergeometric functions developed by Kaneko [28] and Yan [51], to express the expectation value $K_N(s_1,\dots,s_n)$ in the Jacobi and Laguerre β-ensembles as an n-dimensional integral:

$$K_N(s_1,\dots,s_n) = C_N\int_{\mathscr{C}}\cdots\int_{\mathscr{C}}\prod_{j=1}^n e^{-Np(t_j)}\prod_{1\le j<k\le n}|t_j-t_k|^{4/\beta} q_N(t;s)\,dt_1\cdots dt_n. \tag{1.11}$$

where $\mathscr{C}$ and $q_N(t;s)$ respectively denote the circle in the complex plane (or intervals of real numbers) and multivariate hypergeometric function whose precise forms depend on the ensemble.

Such passage from an N-dimensional integral with Dyson parameter β to an n-dimensional integral with Dyson parameter $\beta' = 4/\beta$ is an example of duality relation, which turns out to be very useful since the second n-dimensional integral allows us, in principle at least, to take the limit $N\to\infty$. As observed in [4],

when $s_1 = \cdots = s_n$, the function $q_N(t;s)$ greatly simplifies. This was exploited in [11] for determining the asymptotic behavior of the eigenvalue marginal density $\rho_N(x)$.

Other dualities for the Hermite and Laguerre β-Ensembles were obtained in [10]. One particular duality was used to prove that as $N \to \infty$, for $\beta = 1, 2, 4$, and for an appropriate choice of A and B, the expectation $K_N(A+Bs_1, \ldots, A+Bs_n)$ in the HβE is proportional to the folowing generalized Airy function (see Section 2 for more details about the notation):

$$\mathrm{Ai}^{(\beta/2)}(s_1, \ldots, s_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ip_3(w)/3} |\Delta(w)|^{4/\beta} \;{}_0\mathcal{F}_0^{(\beta/2)}(s_1, \ldots, s_n; iw) d^n w, \tag{1.12}$$

which is absolutely convergent for all $(s_1, \ldots, s_n) \in \mathbb{R}^n$ and $\beta \in \mathbb{R}_+$. The case $\beta = 2$ is proportional to Kontsevich's matrix Airy function [33].

1.3. **Main results.** We prove that at the soft edge, the expectation value of products of characteristic polynomials in both LβE and HβE actually lead to the same multivariate Airy function. Note that the simplest asymptotics for the multivariate Airy function are given in Proposition 3.10.

Theorem 1.1 (Soft edge expectation)**.** *Let*

$$A,\, B = \begin{cases} (2N)^{1/2},\; 2^{-1/2}N^{-1/6} & \text{for the H}\beta\text{E}, \\ 4N, \qquad 2(2N)^{1/3} & \text{for the L}\beta\text{E}. \end{cases} \tag{1.13}$$

Then as $N \to \infty$,

$$\Phi_{N,n}^{-1}\, \varphi_N(A + Bs_1, \ldots, A + Bs_n) \sim (2\pi)^n (\Gamma_{4/\beta,n})^{-1}\, \mathrm{Ai}^{(\beta/2)}(s_1, \ldots, s_n), \tag{1.14}$$

Note that the coefficients $\Phi_{N,n}$ and $\Gamma_{4/\beta,n}$ are given in the appendix.

The above theorem suggests that the multivariate Airy function is the universal expectation value at the soft edge. In other words, for any β-ensemble characterized by a potential V, the average of the product of n characteristic polynomials, when appropriately rescaled and re-centered at the soft edge, should become independent of V and should be proportional to $\mathrm{Ai}^{(\beta/2)}(s_1, \ldots, s_n)$ as $N \to \infty$.

The indication for universality is even stronger in the bulk of the spectrum. We indeed find that the three classical β-ensembles possess the same asymptotic limit for the weighted expectation value φ_N in the bulk, which turns out to be a multivariate hypergeometric function of exponential type. We only state the result in the case of $n = 2m$, for simplicity; for $n = 2m - 1$, the combination of φ_N and φ_{N-1} exhibits a universal pattern, which is given in Theorem 4.1.

Theorem 1.2 (Bulk expectation)**.** *For the* HβE, LβE, *and* JβE, *let A be equal to* $\sqrt{2N}$, $4N$, *and* 1, *respectively. Let*

$$\rho(u) = \begin{cases} \frac{2}{\pi}\sqrt{1-u^2}, & u \in (-1,1), \quad \text{H}\beta\text{E}, \\ \frac{2}{\pi}\sqrt{\frac{1-u}{u}}, & u \in (0,1), \quad \text{L}\beta\text{E}, \\ \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}, & u \in (0,1), \quad \text{J}\beta\text{E}. \end{cases} \tag{1.15}$$

Assume moreover that $n = 2m$ is even. Then as $N \to \infty$,

$$\frac{1}{\Psi_{N,2m}} \varphi_N\Big(Au + \frac{As_1}{\rho(u)N}, \ldots, Au + \frac{As_n}{\rho(u)N}\Big) \sim \gamma_m(4/\beta)\, e^{-i\pi p_1(s)} \;{}_1F_1^{(\beta/2)}(2m/\beta; 2n/\beta; 2i\pi s) \tag{1.16}$$

where $\Psi_{N,2m}$ and $\gamma_m(4/\beta)$ respectively stand for the coefficient given in (A.8) *and* (A.13).

It's worth emphasizing that the universal coefficient $\gamma_m(4/\beta)$, when $\beta = 2$, is conjectured to be closely related to the moments of the Riemann's ζ-function [31].

The hard edge also involves a single hypergeometric series, which is ${}_0F_1^{(\beta/2)}$. The latter can be seen as a multivariate Bessel function. We may thus surmise once more that the asymptotic expectation at the hard edge is universal.

Theorem 1.3 (Hard edge expectation). *Let B be equal to N^{-1} and N^{-2} for the* LβE *and* JβE*, respectively. Then as $N \to \infty$,*

$$\frac{1}{\xi_{N,n}} K_N(Bs_1, \dots, Bs_n) \sim \ {}_0F_1^{(\beta/2)}\left((2/\beta)(\lambda_1 + n); -s_1, \dots, -s_n\right). \tag{1.17}$$

The coefficient $\xi_{N,n}$ is given in (A.10).

Before explaining how we prove these theorems, let us give some immediate consequences. As previously mentioned, if β is even and if $n = 2m = \beta k$, then scaling limits of the k-point correlation functions immediately follow from the above theorems. Since the hard-edge case for the LβE and JβE is already known (see Section 13.2.5 in [23]), we only display below the results for the soft edge and in the bulk.

Corollary 1.4 (Soft edge correlations). *Assume that β is even and $n = \beta k$. Let A and B be as in Theorem 1.1. Then as $N \to \infty$ in the* HβE *and* LβE*,*

$$B^k R_{k,N}(A + Bx_1, \dots, A + Bx_k) \ \sim \ a_k(\beta)\ |\Delta(x)|^\beta \left[\mathrm{Ai}^{(\beta/2)}(s)\right]_{\{s\}\mapsto\{x\}}, \tag{1.18}$$

where $a_k(\beta)$ is given in (A.11).

Corollary 1.5 (Bulk correlations). *Assume that β is even and $n = \beta k$. Let A and $\rho(u)$ be as in Theorem 1.2. Then as $N \to \infty$ in the* HβE, LβE, *and* JβE,

$$\Big(\frac{A}{\rho(u)N}\Big)^k R_{k,N}\Big(Au + \frac{Ax_1}{\rho(u)N}, \dots, Au + \frac{Ax_k}{\rho(u)N}\Big) \ \sim \\ b_k(\beta)\ |\Delta(2\pi x)|^\beta \left[e^{-i\pi p_1(s)}\ {}_1F_1^{(\beta/2)}(n/\beta; 2n/\beta; 2i\pi s)\right]_{\{s\}\mapsto\{x\}}, \tag{1.19}$$

where $b_k(\beta)$ is given in (A.12).

We stress that the function on the right-hand side of (1.19) already appeared in Random Matrix Theory: it is exactly the same as the limiting k-point correlation function of the circular β-ensemble with β even, which is equal to the function $\rho_{(k)}^{\mathrm{bulk}}(x_1, \dots, x_k)$ given in Proposition 13.2.3 of [23]. Moreover, very recently, the limiting k-point correlation function in the bulk of the HβE was shown to be universal (see Theorem 1.2 in [19]). Note that the latter reference does not give however, the explicit form of this universal k-point correlation function. As a consequence of Corollary 1.5, we now know that the universal k-point correlation function, when β is even, is equal to the hypergeometric function on the right-hand side of (1.19).

To the best of our knowledge, there is still no universality theorem regarding the limiting k-point correlation function at the soft edge of the spectrum in β-ensembles. However, assuming that such a universal correlation function exits for β even, we see from Corollary 1.4 that it must be equal to the k-variable function on the right-hand side of (1.18), and as a consequence, it must involve the multivariate Airy function $\mathrm{Ai}^{(\beta/2)}$.

1.4. **Organization of the article and proofs.** As explained in Section 2, the multivariate hypergeometric functions of the form ${}_pF_q^{(\alpha)}$ are defined as series of Jack symmetric polynomials. Although apparently complicated, these series can be evaluated efficiently by simple numerical methods [32].

The proof of Theorem 1.3 becomes trivial once we know that the expectations of products of characteristic polynomials in the LβE and JβE are respectively given by hypergeometric functions of the form ${}_2F_1^{(\beta/2)}$ and ${}_1F_1^{(\beta/2)}$. This is known since [4].

The proof of Theorems 1.1 and 1.2 is not so simple as that of Theorem 1.3. Indeed, the calculation of scaling limits in the bulk and at the soft edge requires the asymptotic evaluation of integrals of Selberg type, such as (1.11). The whole Section 3 is devoted to this task. We generalize the Laplace or steepest descent methods for higher dimensional integrals that contain the absolute value of a Vandermonde determinant.

In Section 4, we finally apply these results to our three classical β-ensembles. Note that these explicit calculations abundantly make use of simple transformations of multivariate hypergeometric series, which will be given in the next section. Some remarks on PDEs satisfied by the scaling limits are given in Section 5.

2. Jack polynomials and hypergeometric functions

This section first provides a brief review of some aspects of symmetric polynomials and especially Jack polynomials. The classical references on the subject are Macdonald's book [37] and Stanley's article [49]. This will allow us to introduce the multivariate hypergeometric functions [28,34,51]. A few results proved here will be used later in the article.

2.1. **Partitions.**

A partition $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots)$ is a sequence of non-negative integers κ_i such that

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_i \geq \cdots$$

and only a finite number of the terms κ_i are non-zero. The number of non-zero terms is referred to as the length of κ, and is denoted $\ell(\kappa)$. We shall not distinguish between two partitions that differ only by a string of zeros. The weight of a partition κ is the sum

$$|\kappa| := \kappa_1 + \kappa_2 + \cdots$$

of its parts, and its diagram is the set of points $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \kappa_i$. Reflection in the diagonal produces the conjugate partition $\kappa' = (\kappa_1', \kappa_2', \ldots)$.

The set of all partitions of a given weight are partially ordered by the dominance order: $\kappa \leq \sigma$ if and only if $\sum_{i=1}^k \kappa_i \leq \sum_{i=1}^k \sigma_i$ for all k. One easily verifies that $\kappa \leq \sigma$ if and only if $\sigma' \leq \kappa'$.

2.2. **Jack polynomials.**

Let $\Lambda_n(x)$ denote the algebra of symmetric polynomials in n variables $x_1, \ldots, x_n$ and with coefficients in the field $\mathbb{F}$. In this article, $\mathbb{F}$ is assumed to be the field of rational functions in the parameter α. As a ring, $\Lambda_n(x)$ is generated by the power-sums:

$$p_k(x) := x_1^k + \cdots + x_n^k. \tag{2.1}$$

The ring of symmetric polynomials is naturally graded: $\Lambda_n(x) = \oplus_{k\geq 0}\Lambda_n^k(x)$, where $\Lambda_n^k(x)$ denotes the set of homogeneous polynomials of degree k. As a vector space, $\Lambda_n^k(x)$ is equal to the span over $\mathbb{F}$ of all symmetric monomials $m_\kappa(x)$, where κ is a partition of weight k and

$$m_\kappa(x) := x_1^{\kappa_1} \cdots x_n^{\kappa_n} + \text{distinct permutations}.$$

Note that if the length of the partition κ is larger than n, we set $m_\kappa(x) = 0$.

The whole ring $\Lambda_n(x)$ is invariant under the action of homogeneous differential operators related to the Calogero-Sutherland models [4]:

$$E_k = \sum_{i=1}^n x_i^k \frac{\partial}{\partial x_i}, \qquad D_k = \sum_{i=1}^n x_i^k \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1\leq i\neq j\leq n} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i}, \qquad k \geq 0. \tag{2.2}$$

The operators E_1 and D_2 are special since they also preserve each $\Lambda_n^k(x)$. They can be used to define the Jack polynomials. Indeed, for each partition κ, there exists a unique symmetric polynomial $P_\kappa^{(\alpha)}(x)$ that satisfies the two following conditions [49]:

$$(1) \qquad P_\kappa^{(\alpha)}(x) = m_\kappa(x) + \sum_{\mu<\kappa} c_{\kappa\mu} m_\mu(x) \qquad \text{(triangularity)} \tag{2.3}$$

$$(2) \qquad \Big(D_2 - \frac{2}{\alpha}(n-1)E_1\Big) P_\kappa^{(\alpha)}(x) = \epsilon_\kappa P_\kappa^{(\alpha)}(x) \qquad \text{(eigenfunction)} \tag{2.4}$$

where the coefficients ϵ_κ and $c_{\kappa\mu}$ belong to $\mathbb{F}$. Because of the tiangularity condition, $\Lambda_n(x)$ is also equal to the span over $\mathbb{F}$ of all Jack polynomials $P_\kappa^{(\alpha)}(x)$, with κ a partition of length less or equal to n.

2.3. **Hypergeometric series.** Recall that the arm-lengths and leg-lengths of the box (i,j) in the partition κ are respectively given by

$$a_\kappa(i,j) = \kappa_i - j \qquad \text{and} \qquad l_\kappa(i,j) = \kappa'_j - i. \tag{2.5}$$

We define the hook-length of a partition κ as the following product:

$$h_\kappa^{(\alpha)} = \prod_{(i,j)\in\kappa} \Big(1 + a_\kappa(i,j) + \frac{1}{\alpha} l_\kappa(i,j)\Big). \tag{2.6}$$

Closely related is the following α-deformation of the Pochhammer symbol:

$$[x]_\kappa^{(\alpha)} = \prod_{1\le i\le \ell(\kappa)} \Big(x - \frac{i-1}{\alpha}\Big)_{\kappa_i} = \prod_{(i,j)\in\kappa} \Big(x + a'_\kappa(i,j) - \frac{1}{\alpha} l'_\kappa(i,j)\Big) \tag{2.7}$$

In the middle of the last equation, $(x)_j \equiv x(x+1)\cdots(x+j-1)$ stands for the ordinary Pochhammer symbol, to which $[x]_\kappa^{(\alpha)}$ clearly reduces for $\ell(\kappa) = 1$. The right-hand side of (2.7) involves the co-arm-lengths and co-leg-lengths box (i,j) in the partition κ, which are respectively defined as

$$a'_\kappa(i,j) = j - 1, \qquad \text{and} \qquad l'_\kappa(i,j) = i - 1. \tag{2.8}$$

We are now ready to give the precise definition of the hypergeometric series used in the article.

Definition 2.1. Fix $p,q \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and let $a_1,\ldots,a_p,b_1,\ldots,b_q$ be complex numbers such that $(i-1)/\alpha - b_j \notin \mathbb{N}_0$ for all $i \in \mathbb{N}_0$. We then define the (p,q)-type hypergeometric series as follows [28,34,51]:

$${}_pF_q^{(\alpha)}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{k=0}^\infty \sum_{|\kappa|=k} \frac{1}{h_\kappa^{(\alpha)}} \frac{[a_1]_\kappa^{(\alpha)}\cdots[a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)}\cdots[b_q]_\kappa^{(\alpha)}} P_\kappa^{(\alpha)}(x). \tag{2.9}$$

Similarly the hypergeometric series in two sets of n variables, $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$, is given in [51] by

$${}_p\mathcal{F}_q^{(\alpha)}(a_1,\ldots,a_p;b_1,\ldots,b_q;x;y) = \sum_\kappa \frac{1}{h_\kappa^{(\alpha)}} \frac{[a_1]_\kappa^{(\alpha)}\cdots[a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)}\cdots[b_q]_\kappa^{(\alpha)}} \frac{P_\kappa^{(\alpha)}(x)P_\kappa^{(\alpha)}(y)}{P_\kappa^{(\alpha)}(1^n)}, \tag{2.10}$$

where we have used the shorthand notation 1^n for $\overbrace{1,\ldots,1}^{n}$.

Note that when $p \le q$, the above series converge absolutely for all $x \in \mathbb{C}^n$, $y \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}_+$. In the case where $p = q+1$, then the series converge absolutely for all $\|x\| < 1$, $\|y\| < 1$ and $\alpha \in \mathbb{R}_+$. See [28] for more details about convergence issues.

Now we give some translation properties of ${}_0\mathcal{F}_0^{(\alpha)}$ and ${}_1\mathcal{F}_0^{(\alpha)}$, which prove to be of practical importance. For convenience, we write

$$(a^n) = (\overbrace{a,\ldots,a}^{n}), \qquad b + ax = (b+ax_1,\ldots,b+ax_n), \qquad \frac{x}{1-ax} = \Big(\frac{x_1}{1-ax_1},\ldots,\frac{x_n}{1-ax_n}\Big), \tag{2.11}$$

where a,b are complex numbers and $x = (x_1,\ldots,x_n)$.

Proposition 2.2. *We have*

$${}_0\mathcal{F}_0^{(\alpha)}(a+x;b+y) = \exp\{nab + ap_1(y) + bp_1(x)\}\,{}_0\mathcal{F}_0^{(\alpha)}(x;y) \tag{2.12}$$

and

$${}_1\mathcal{F}_0^{(\alpha)}(a;b+x;y) = \prod_{j=1}^n (1-by_j)^{-a}\,{}_1\mathcal{F}_0^{(\alpha)}(a;x;\frac{y}{1-by}). \tag{2.13}$$

Proof. First set $F(a,b) = {}_0\mathcal{F}_0^{(\alpha)}(a+x;b+y)$. Then, according to Eq. (3.3) of [4],

$$E_0^{(y)}\,{}_0\mathcal{F}_0^{(\alpha)}(x;y) = p_1(x)\,{}_0\mathcal{F}_0^{(\alpha)}(x;y), \tag{2.14}$$

so that

$$\frac{\partial F}{\partial b} = E_0^{(y)}\,{}_0\mathcal{F}_0^{(\alpha)}(x;y)\big|_{x\to a+x,y\to b+y} = p_1(a+x)\,F.$$

Similarly,

$$\frac{\partial F}{\partial a} = p_1(b+y)\,F.$$

By solving those differential equations we get

$$F(a,b) = e^{ap_1(b+y)}F(0,b) = e^{ap_1(b+y)}e^{bp_1(x)}F(0,0),$$

which is the first desired result.

For the second result, it suffices to prove

$$ {}_1\mathcal{F}_0^{(\alpha)}(a;x;y)\prod_{j=1}^{n}(1-by_j)^a = {}_1\mathcal{F}_0^{(\alpha)}(a;x-b;\frac{y}{1-by}). \tag{2.15}$$

Now let $G_l(b)$ and $G_r(b)$ respectively denote the left-hand and right-hand sides of (2.15).

In Eq. (A.1) of [4], we substitute x and y by cx and y/b, respectively. Then we let $b,c\to\infty$, and conclude that ${}_1\mathcal{F}_0^{(\alpha)}(a;x;y)$ satisfies

$$E_0^{(x)}F - E_2^{(y)}F = ap_1(y)F. \tag{2.16}$$

From (2.16), we get

$$G_r'(b) = \Big(-E_0^{(x)}\,{}_1\mathcal{F}_0^{(\alpha)}(a;x;y) + E_2^{(y)}\,{}_1\mathcal{F}_0^{(\alpha)}(a;x;y)\Big)\Big|_{x\to x-b,y\to\frac{y}{1-by}} = -ap_1(\frac{y}{1-by})\,G_r(b).$$

Finally, it's easy to check that $G_l(b)$ also satisfies the differential equation just given above with the same initial condition at $b=0$, which means $G_l(b) = G_r(b)$. □

Corollary 2.3.

$$ {}_0\mathcal{F}_0^{(\alpha)}(x_1,\ldots,x_n;a^k,b^{n-k}) = e^{bp_1(x)}\,{}_1F_1^{(\alpha)}(k/\alpha;n/\alpha;(a-b)x_1,\ldots,(a-b)x_n) \tag{2.17}$$

$$= e^{(a-b)kx_1+bp_1(x)}\,{}_1F_1^{(\alpha)}(k/\alpha;n/\alpha;(a-b)(x_2-x_1),\ldots,(a-b)(x_n-x_1)). \tag{2.18}$$

Proof. Proposition 2.2 implies that

$$ {}_0\mathcal{F}_0^{(\alpha)}(x_1,\ldots,x_n;a^k,b^{n-k}) = e^{bp_1(x)}\,{}_0\mathcal{F}_0^{(\alpha)}(x_1,\ldots,x_n;(a-b)^k,0^{n-k}).$$

The right-hand side is equal to

$$e^{bp_1(x)}\sum_{\ell(\kappa)\le k}\frac{(a-b)^{|\kappa|}}{h_\kappa^{(\alpha)}}\frac{P_\kappa^{(\alpha)}(x)P_\kappa^{(\alpha)}(1^k)}{P_\kappa^{(\alpha)}(1^n)}.$$

But we also know from Eq. (10.20) of Chapter VI [37] and Eq. (2.7) above, that

$$P_\kappa^{(\alpha)}(1^k) = \frac{\alpha^{|\kappa|}[k/\alpha]_\kappa^{(\alpha)}}{\prod_{(i,j)\in\kappa}(\alpha a_\kappa(i,j)+l_\kappa(i,j)+1)}.$$

Moreover, one easily checks that $[k/\alpha]_\kappa^{(\alpha)} = 0$ whenever $\ell(\kappa) > k$. Consequently,

$$ {}_0\mathcal{F}_0^{(\alpha)}(x_1,\ldots,x_n;a^k,b^{n-k}) = e^{bp_1(x)}\sum_\kappa\frac{(a-b)^{|\kappa|}}{h_\kappa^{(\alpha)}}\frac{[k/\alpha]_\kappa^{(\alpha)}}{[n/\alpha]_\kappa^{(\alpha)}}P_\kappa^{(\alpha)}(x),$$

which is equivalent to the first equality.

Likewise, the second equality also follows from Proposition 2.2. □

We conclude this section with introduction of the following symbol, for simplicity,

$$E_j^{(\alpha)}(x) := E_j^{(\alpha)}(x_1, \ldots, x_n) = {}_0\mathcal{F}_0^{(\alpha)}(x; (-1)^j, 1^{n-j}), \qquad j = 0, 1, \ldots, n. \tag{2.19}$$

These functions possess some similar properties with the exponential function, for instance,

$$E_j^{(\alpha)}(2i\pi + x) = E_j^{(\alpha)}(x).$$

3. Asymptotic methods for integrals of Selberg type

We want to get the large N asymptotic behaviour for integrals of the form (1.11) by generalizing the classical steepest descent method, or more generally Laplace's method for contour integrals. We refer the reader to Olver's textbook [44] for more details on Laplace's method in the one-dimensional case.

Before going any further, let us adopt some new notational conventions. Our variables are $t = (t_1, \ldots, t_n)$ and the Vandermonde determinant is given by

$$\Delta(t) = \prod_{1 \le i < j \le n} (t_i - t_j). \tag{3.1}$$

Moreover, $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes a sequence of parameters such that $\lambda_j > 0$ for all j while $k = (k_1, \ldots, k_n)$ denotes a sequence of non-negative integers of weight $|k| = \sum_j k_j$. Finally, $t^{\lambda-1} = \prod_j t_j^{\lambda_j - 1}$ and $t^k = \prod_j t_j^{k_j}$.

3.1. Watson's Lemma.

We first give the Watson's lemma for multiple integrals with Vandermonde determinants. The process of the proof will suggest a natural extension to contour integrals.

Lemma 3.1. *Let $\|t\| = \max\{|t_1|, \ldots, |t_n|\}$ and $q(t)$ be a function of the positive real variables t_j, such that*

$$q(t) = t^{\lambda-1}\Big(\sum_{j=0}^{m-1} a_j(t) + O(\|t\|^m)\Big) \quad (\|t\| \to 0) \tag{3.2}$$

where

$$a_j(t) = \sum_{|k|=j} a_{j,k}\, t^k$$

is a homogenous polynomial of degree j. Then

$$I_N := \int_0^\infty \cdots \int_0^\infty e^{-N\sum_{j=1}^n t_j} |\Delta(t)|^\beta q(t)\, dt_1 \cdots dt_n = \sum_{j=0}^{m-1} \frac{A_j}{N^{n_\beta + |\lambda| + j}} + O(N^{-n_\beta - |\lambda| - m}) \tag{3.3}$$

as $N \to \infty$ provided that this integral converges throughout its range for all sufficiently large N, where $n_\beta = \beta n(n-1)/2$ and

$$A_j = \int_0^\infty \cdots \int_0^\infty t^{\lambda-1} e^{-\sum_{j=1}^n t_j} |\Delta(t)|^\beta a_j(t)\, dt_1 \cdots dt_n. \tag{3.4}$$

Proof. For each m, define

$$\phi_m(t) = q(t) - t^{\lambda-1} \sum_{j=0}^{m-1} a_j(t). \tag{3.5}$$

One obtains

$$I_N = \sum_{j=0}^{m-1} \frac{A_j}{N^{n_\beta + |\lambda| + j}} + \int_0^\infty \cdots \int_0^\infty e^{-N\sum_{j=1}^n t_j} |\Delta(t)|^\beta \phi_m(t)\, dt_1 \cdots dt_n. \tag{3.6}$$

As $\|t\| \to 0$ we have $\phi_m(t) = t^{\lambda-1} O(\|t\|^m)$. This means that there exist positive constants k_m and K_m such that

$$|\phi_m(t)| \le K_m \|t\|^{m + |\lambda| - n} \quad (0 < \|t\| \le k_m).$$

Accordingly,

$$\Big|\int_0^{k_m}\cdots\int_0^{k_m} e^{-N\sum_{j=1}^n t_j}|\Delta(t)|^\beta \phi_m(t)\,dt_1\cdots dt_n\Big| < \frac{C_m K_m}{N^{n_\beta+|\lambda|+m}}, \tag{3.7}$$

where

$$C_m = \int_0^\infty\cdots\int_0^\infty e^{-\sum_{j=1}^n t_j}|\Delta(t)|^\beta \|t\|^{|\lambda|+m-n}\,dt_1\cdots dt_n.$$

For the contribution from the range

$$(0,\infty)^n\backslash(0,k_m)^n = \bigcup(J_1\times\cdots\times J_n),$$

where $J_j=(0,k_m)$ or $[k_m,\infty)$ but at least one of which is the infinite interval $[k_m,\infty)$. One knows there are (2^n-1) intervals in the union above. Without loss of generality, we only consider the case when $J=(0,k_m)^{n-1}\times[k_m,\infty)$.

Let N_0 be a value of N for which the integral on the right-hand side of (3.6) exists, and write

$$\Phi_m(t) = \int_0^{t_1}\cdots\int_0^{t_{n-1}}\int_{k_m}^{t_n} e^{-N_0\sum_{j=1}^n v_j}|\Delta(t)|^\beta\phi_m(v)\,dv_1\cdots dv_n,$$

so that $\Phi_m(t)$ is continuous and bounded in $J=(0,k_m)^{n-1}\times[k_m,\infty)$. Let L_m denote the supremum of $|\Phi_m(t)|$ in this range. When $N>N_0$, one finds by partial integration

$$\begin{aligned}
I_J :&= \int_J e^{-N\sum_{j=1}^n t_j}|\Delta(t)|^\beta\phi_m(t)\,dt_1\cdots dt_n\\
&= \int_J e^{-(N-N_0)\sum_{j=1}^n t_j}e^{-N_0\sum_{j=1}^n t_j}|\Delta(t)|^\beta\phi_m(t)\,dt_1\cdots dt_n\\
&= \sum_{j=0}^{n-1}(N-N_0)^{j+1}\sum_{1\le i_1<\cdots<i_j\le n-1}\int_{(0,k_m)^j\times[k_m,\infty)}\Big[e^{-(N-N_0)\sum_{l=1}^n t_l}\Phi_m(t)\Big]dt_{i_1}\cdots dt_{i_j}dt_n,
\end{aligned} \tag{3.8}$$

where $[f(t)]$ denotes the evaluation of $f(t)$ at $t_l=k_m$ if $l\neq i_1,\ldots,i_j,n$.

Thus

$$\begin{aligned}
|I_J| &\le L_m\sum_{j=0}^{n-1}\binom{n-1}{j}(N-N_0)^{j+1}e^{-(n-j-1)(N-N_0)k_m}\int_{(0,k_m)^j\times[k_m,\infty)}e^{-(N-N_0)\sum_{l=n-j}^n t_l}dt_{i_{n-j}}\cdots dt_n\\
&= L_m e^{-(N-N_0)k_m}.
\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we immediately see that the integral on the right-hand side of (3.6) is $O(N^{-n_\beta-|\lambda|-m})$ as $N\to\infty$, and the lemma is proved. □

Corollary 3.2. *With the same assumptions as in the previous theorem and with $\lambda_i=\mu>0$ for all i, we have as $N\to\infty$,*

$$\int_{(0,\infty)^n} e^{-N\sum_{j=1}^n t_j}|\Delta(t)|^\beta q(t)\,d^nt \;=\; \prod_{j=0}^{n-1}\frac{\Gamma(1+\beta/2+j\beta/2)\Gamma(\mu+j\beta/2)}{\Gamma(1+\beta/2)}\frac{a_0}{N^{n_\beta+\mu n}}\;+O(N^{-n_\beta-\mu n-1})$$

where

$$a_0=\left[t^{-\lambda+1}q(t)\right]_{t=(0,\ldots,0)}.$$

Remark 3.3. The integrals A_j, given by (3.4), is in general very difficult to evaluate. However, if $q(t)$ is a symmetric function and $\lambda_1=\cdots=\lambda_n$, then A_j can be calculated by the Macdonald-Kadell-Selberg integrals, see [37], p. 386 (a) or the original papers [27, 28].

3.2. **Laplace's method: a single saddle point.** Consider the integral

$$I_N = \int_a^b \cdots \int_a^b \exp\{-N \sum_{j=1}^{n} p(t_j)\} h(\Delta(t))\, q(t)\, dt_1 \cdots dt_n, \tag{3.10}$$

in which the path $\mathscr{P}$ between the points a and b, say, is a contour in the complex plane $\mathbb{C}$, $p(x)$ and $q(t)$ are analytic functions of x and $t = (t_1, \ldots, t_n)$ in the domains $\mathbf{T} \subseteq \mathbb{C}$ and $\mathbf{T}^n$, respectively. Here N is a positive parameter and $h(x)$ is a homogeneous analytic function of degree $\nu \geq 0$, i.e., $h(cx) = c^\nu h(x), \forall\ \mathrm{Re}\, c > 0$.

Like the one-dimensional integral, to obtain the asymptotics of integrals one usually needs to deform the path through some special points at which $p'(x) = 0$, called *saddle points*, we refer to sections 7 and 10, [44] for more details about *saddle point* and paths of *steepest descent*.

Recall that x_0 is a saddle point of order $\mu - 1$ if

$$p'(x_0) = \cdots = p^{(\mu-1)}(x_0) = 0,\ p^{(\mu)}(x_0) \neq 0,$$

where the integer $\mu \geq 2$. In particular, when $\mu = 2$ it is called a simple saddle point. The most common cases for the integrals are that $p(x)$ has (1) one simple saddle point; (2) two simple saddle points; (3) one saddle point of order 2, at interior points of the integration path. Those occur ubiquitously in Random Matrix Theory, corresponding to the hard-edge, bulk and soft-edge limiting behavior. Fortunately, deformations to the path through *saddle points* for our multi-dimensional integrals can reduce to the one-dimensional case. But great care must be taken in dealing with the part involving Vandermonde determinant.

We will first consider the case of a single saddle point of order $\mu - 1$, which generalizes both cases (1) and (3). By convention $\mathrm{ph}(z)$ denotes the phase or argument of complex variable z. For an interior point x_0 of $(a, b)_{\mathscr{P}}$, we denote

$$\omega = \text{angle of slope of } \mathscr{P} \text{ at } x_0 = \lim\{\mathrm{ph}(x - x_0)\} \qquad (x \to x_0 \text{ along } (x_0, b)_{\mathscr{P}}). \tag{3.11}$$

We moreover assume the following:

(i) $p(x)$ and $q(t)$ are single-valued and holomorphic in the domains $\mathbf{T} \subseteq \mathbb{C}$ and $\mathbf{T}^n$; $h(x)$ is a homogeneous analytic function of degree $\nu \geq 0$ in $\mathbb{C}$.

(ii) The integration path $\mathscr{P}$ is independent of N. The endpoints a and b of $\mathscr{P}$ are finite or infinite, and $(a, b)_{\mathscr{P}}$ lies within $\mathbf{T}$.

(iii) $p'(x)$ has exactly one zero of order $\mu - 1$ at an interior point x_0 of $\mathscr{P}$, i.e., $p'(x_0) = \cdots = p^{(\mu-1)}(x_0) = 0, p^{(\mu)}(x_0) \neq 0$; in the neighborhoods of x_0 and $t_0 = (x_0, \ldots, x_0)$, $p(x)$ and $q(t)$ can be expanded in convergent series of the form

$$p(x) = p(x_0) + \sum_{s=0}^{\infty} p_s (x - x_0)^{s+\mu}, q(t) = (t - t_0)^{\lambda - 1} \sum_{j=0}^{\infty} q_j (t - t_0),$$

where $p_0 \neq 0$ and $q_j(t) = \sum_{|k|=j} q_{j,k}\, t^k$.

(iv) There exists $N_0 > 0$ such that I_{N_0} converges absolutely at $(a, \ldots, a)$ and $(b, \ldots, b)$.

(v) $\mathrm{Re}\{p(x) - p(x_0)\}$ is positive on $(a, b)_{\mathscr{P}}$, except at x_0, and is bounded away from zero as $x \to a$ or b along $\mathscr{P}$.

Remark 3.4. Let $p_0 := p^{(\mu)}(x_0)/\mu!$. When the phase of $(x - x_0)^\mu p_0$ is exactly equal to zero, then the first non-null term in $\mathrm{Re}\{p(x) - p(x_0)\}$ reaches its minimum value while $\mathrm{Im}\{p(x) - p(x_0)\}$ gets equal to zero. Any path $\mathscr{P}$ for x that guarantees $\mathrm{ph}\{(x - x_0)^\mu p_0\} = 0$ is called the steepest descents. For our special cases coming from Random Matrix Theory, we will always be able to deform a part of the contour of integration to one of steepest descent contours. For the general case, however, the condition $\mathrm{Re}\{p(x) - p(x_0)\} > 0$ is sufficient.

We proceed in three steps.

Step 1. We introduce the following convention: the value of $\omega_0 = \mathrm{ph}(p_0)$ is not necessary the principal one, but is chosen to satisfy

$$|\omega_0 + \mu\, \omega| \leq \frac{1}{2}\pi, \tag{3.12}$$

and the branch of $\mathrm{ph}(p_0)$ is used in constructing all fractional powers of p_0 which occur. For example, $p_0^{1/\mu}$ means $\exp\{(\ln|p_0| + i\omega_0)/\mu\}$. Since

$$p(x) - p(x_0) \sim p_0(x-x_0)^\mu, \qquad \text{as } x \to x_0 \text{ along } (a,b)_{\mathscr{P}},$$

introduce new variables w_j by the equation

$$w_j^\mu = (p(t_j) - p(x_0))/p_0. \tag{3.13}$$

The branch of $\mathrm{ph}(w_j)$ is determined by

$$\mu\,\mathrm{ph}(w_j) \to \mu\omega \qquad (\, w_j \to x_0 \ \text{ along } \ (x_0,b)_{\mathscr{P}}\,), \tag{3.14}$$

and by continuity elsewhere.

For small $\|t - t_0\|$, Condition (iii) and the Binomial theorem yield

$$w_j = (t_j - x_0)\{1 + \frac{p_1}{\mu p_0}(t_j - x_0) + \cdots\}.$$

Application of the inversion theorem for analytic functions shows that for all sufficiently small $\rho > 0$, the disk $|t_j - x_0| < \rho$ is mapped conformally on a domain $\mathbf{W}$ containing $w_j = 0$. Moreover, if $w_j \in \mathbf{W}$ then $t_j - x_0$ can be expanded in a convergent series

$$t_j - x_0 = \sum_{s=1}^{\infty} c_s w_j^s,$$

in which the coefficients c_s are expressible in terms of the p_s. For example, $c_1 = 1$ and $c_2 = -p_1/(\mu p_0)$.

Let τ_1, τ_2 be points of $(a,x_0)_{\mathscr{P}}, (x_0,b)_{\mathscr{P}}$ respectively chosen sufficiently close to x_0 to ensure that the disk

$$|w_j| \le \min\{|p(\tau_1) - p(x_0)|^{1/\mu}, |p(\tau_2) - p(x_0)|^{1/\mu}\}$$

is contained in $\mathbf{W}$. Then $(\tau_1, x_0)_{\mathscr{P}}$ and $(x_0, \tau_2)_{\mathscr{P}}$ may be deformed to make its w_j map straight lines L_{11} and L_{21}, respectively. If

$$\kappa_j = (p(\tau_j) - p(x_0))/p_0, \quad j = 1, 2,$$

then L_{11} and L_{21} are directed line segments, respectively, from κ_1 to 0 and from 0 to κ_2. On the other hand, let $L_1 = L_{12} \cup L_{11}$ and $L_2 = L_{21} \cup L_{22}$ denote the half-lines containing points $\infty, \kappa_1, 0$ and $0, \kappa_2, \infty$, respectively. Thus we have

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \cdots \int_{\tau_1}^{\tau_2} \exp\{-N\sum_{j=1}^{n} p(t_j)\} h(\Delta(t))\, q(t)\, dt_1 \cdots dt_n = \\ e^{-nNp(x_0)} \int_{L_{11}\cup L_{21}} \cdots \int_{L_{11}\cup L_{21}} \exp\{-Np_0 \sum_{j=1}^{n} w_j^\mu\}\, h(\Delta(w)) f(w)\, dw_1 \cdots dw_n,\end{aligned} \tag{3.15}$$

where

$$f(w) = q(t)\Big(\prod_{i<j}\sum_{s\ge1} c_s \frac{w_i^s - w_j^s}{w_i - w_j}\Big)^\nu \prod_{j=1}^{n} \big(\sum_{s\ge1} s\, c_s w_j^{s-1}\big).$$

For small $\|w\|$, $f(w)$ has a convergent expansion of the form

$$f(w) = w^{\lambda-1}\sum_{j=0}^{\infty} a_j(w), \quad a_j(w) = \sum_{|k|=j} a_{j,k}\, w^k, \tag{3.16}$$

in which the coefficients $a_{j,k}$ can be computed in terms of p_s and $q_{j,k}$. In particular, $a_0 = q_0$.

Step 2. Following the approach for the Watson's lemma, we define $f_m(w), m = 0, 1, \ldots,$ by

$$f(w) = w^{\lambda-1}\sum_{j=0}^{m-1} a_j(w) + w^{\lambda-1} f_m(w). \tag{3.17}$$

Then $f_m(w) = O(\|w\|^m)$. Set $n_\nu = \nu n(n-1)/2$ and

$$A_j = \int_{L_1\cup L_2}\cdots\int_{L_1\cup L_2} \exp\{-p_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, w^{\lambda-1} a_j(w)\, dw_1\cdots dw_n, \tag{3.18}$$

the integral on the right-hand side of (3.15) is rearranged in the form

$$\int_{L_{11}\cup L_{21}}\cdots\int_{L_{11}\cup L_{21}} \exp\{-Np_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, f(w)\, dw_1\cdots dw_n$$
$$= \sum_{j=0}^{m-1} \frac{A_j}{N^{(n_\nu+|\lambda|+j)/\mu}} - \varepsilon_{m,1}(N) + \varepsilon_{m,2}(N). \tag{3.19}$$

Here

$$\varepsilon_{m,1}(N) = \sum_{j=0}^{m-1}\sum_J \int_J \exp\{-Np_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, w^{\lambda-1} a_j(w)\, dw_1\cdots dw_n, \tag{3.20}$$

summed over $J = J_1\times\cdots\times J_n$, $J_j = L_{11}\cup L_{21}$ or $L_{12}\cup L_{22}$, but at least one of which is $L_{12}\cup L_{22}$, and

$$\varepsilon_{m,2}(N) = \int_{L_{11}\cup L_{21}}\cdots\int_{L_{11}\cup L_{21}} \exp\{-Np_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, w^{\lambda-1} f_m(w)\, dw_1\cdots dw_n. \tag{3.21}$$

For $\varepsilon_{m,2}(N)$, splitting the integral domain into 2^n parts, for one of which, for instance, the domain $L_{11}\times\cdots\times L_{11}\times L_{21}$, substitute $w_j = \kappa_1 v_j, j<n$ and $w_n = \kappa_2 v_n$, and note that Condition (v) implies

$$\mathrm{Re}\{p_0\kappa_1^\mu\} \ge \eta_1, \quad \mathrm{Re}\{p_0\kappa_2^\mu\} \ge \eta_1 \tag{3.22}$$

for some positive η_1, we have

$$\int_{L_{11}}\cdots\int_{L_{21}} \exp\{-Np_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, w^{\lambda-1} f_m(w)\, dw_1\cdots dw_n = \int_0^1\cdots\int_0^1$$
$$\exp\{-Np_0(\kappa_1^\mu v_1^\mu + \cdots + \kappa_1^\mu v_{n-1}^\mu + \kappa_2^\mu v_n^\mu\} h(\Delta(w))\, w^{\lambda-1} O(\|w\|^m)\, dv_1\cdots dv_n = O(N^{-(n_\nu+|\lambda|+m)/\mu}).$$

For $\varepsilon_{m,1}(N)$, without loss of generality, consider the case when $J_1 = \cdots = J_{n-1} = L_{11}\cup L_{21}, J_n = L_{12}\cup L_{22}$. Then

$$I_J := \int_J \exp\{-Np_0\sum_{j=1}^n w_j^\mu\} h(\Delta(w))\, w^{\lambda-1} a_j(w)\, dw_1\cdots dw_n$$
$$= \int_J \exp\{-(N-N_0)p_0\sum_{j=1}^n w_j^\mu\} \exp\{-N_0p_0\sum_{j=1}^n w_j^\mu\}\, h(\Delta(w))\, w^{\lambda-1} a_j(w)\, dw_1\cdots dw_n. \tag{3.23}$$

Condition (v) implying that $\mathrm{Re}\{p_0 w_j^\mu\} \ge 0, 1\le j<n$ for $w_j \in L_{11}\cup L_{21}$, combining with (3.22) and Condition (iv) we obtain

$$|I_J| \le O(1)\, e^{-(N-N_0)\eta_1}. \tag{3.24}$$

The combination of the results of this step with (3.15) leads to

$$\int_{\tau_1}^{\tau_2}\cdots\int_{\tau_1}^{\tau_2} \exp\{-N\sum_{j=1}^n p(t_j)\} h(\Delta(t))\, q(t)\, dt_1\cdots dt_n =$$
$$e^{-nNp(x_0)}\Big\{\sum_{j=0}^{m-1}\frac{A_j}{N^{(n_\nu+|\lambda|+j)/\mu}} + O(\frac{1}{N^{(n_\nu+|\lambda|+m)/\mu}})\Big\}, \tag{3.25}$$

as $N\to\infty$.

Step 3. It remains to consider the tail of the integral, that is, the contribution from $(a,b)^n \setminus (\tau_1,\tau_2)^n = \bigcup_J J_1 \times \cdots \times J_n$, $J_j = (\tau_1,\tau_2)$ or $(a,\tau_1)\cup(\tau_2,b)$ but at least one of which is $(a,\tau_1)\cup(\tau_2,b)$. Without loss of generality, we assume that $J_1 = \cdots = J_{n-1} = (\tau_1,\tau_2)$ and $J_n = (a,\tau_1)\cup(\tau_2,b)$. From Condition (v) we know that

$$\mathrm{Re}\{p(x)-p(x_0)\} \geq 0 \,(x \in (a,b)_{\mathscr{P}}) \quad \text{and} \quad \mathrm{Re}\{p(x)-p(x_0)\} \geq \eta_2 \,(x \in (a,\tau_1)_{\mathscr{P}} \cup (\tau_2,b)_{\mathscr{P}}) \tag{3.26}$$

for some $\eta_2 > 0$. Therefore,

$$N\mathrm{Re}\{p(t_n)-p(x_0)\} \geq (N-N_0)\eta_2 + N_0\mathrm{Re}\{p(t_n)-p(x_0)\},$$

and condition (iv) shows

$$\Big|\int_J \exp\{-N\sum_{j=1}^n (p(t_j)-p(x_0))\}h(\Delta(t))\,q(t)\,dt_1\cdots dt_n\Big| \leq$$

$$e^{-(N-N_0)\eta_2}\big|e^{-nN_0p(x_0)}\big|\int_J\Big|\exp\{-N_0\sum_{j=1}^n p(t_j)\}h(\Delta(t))\,q(t)\,dt_1\cdots dt_n\Big|. \tag{3.27}$$

Thus the asymptotic expansion (3.25) is unaffected by the tail integrals.

By following the steps 1-3, we have established the following fundamental result:

Theorem 3.5. *With the foregoing assumptions (i)–(v), $\forall m \in \mathbb{N}$, we have*

$$\int_a^b\cdots\int_a^b \exp\{-N\sum_{j=1}^n p(t_j)\}h(\Delta(t))\,q(t)\,dt_1\cdots dt_n = e^{-nNp(x_0)}\Big\{\sum_{j=0}^{m-1}\frac{A_j}{N^{(n_\nu+|\lambda|+j)/\mu}} + O(\frac{1}{N^{(n_\nu+|\lambda|+m)/\mu}})\Big\},$$

as $N\to\infty$. Here the coefficients A_j are given by (3.18).

Remark 3.6. If we focus on the leading term with the coefficient A_0, then there is an immediate generalization of the previous theorem which will be used later in the article. Let $g(t)$ be analytic in the whole $\mathbb{C}^n$. Now in the integral (3.10), replace $q(t)$ by $q(t)g(N^{1/\bar{\mu}}(t-t_0))$, and assume $\bar{\mu}\geq\mu$. At first sight, the factor $g(N^{1/\bar{\mu}}(t-t_0))$ may become very large as $N\to\infty$. However, by repeating the analysis done in steps 2 and 3 (and paying a particular attention to the rescaling of the variables), we see that $g(N^{1/\bar{\mu}}(t-t_0))$ does not lead to a new significant contribution to the leading coefficient. Most importantly, if $\bar{\mu}=\mu$, then we get a very simple formula:

$$\int_a^b\cdots\int_a^b \exp\{-N\sum_{j=1}^n p(t_j)\}h(\Delta(t))\,q(t)\,g(N^{1/\mu}(t-t_0))\,dt_1\cdots dt_n$$

$$= e^{-nNp(x_0)}\Big\{\frac{A_0'}{N^{(n_\nu+|\lambda|)/\mu}} + O(N^{-(n_\nu+|\lambda|+1)/\mu})\Big\},$$

where

$$A_0' = q_0\int_{L_1\cup L_2}\cdots\int_{L_1\cup L_2}\exp\{-p_0\sum_{j=1}^n w_j^\mu\}h(\Delta(w))\,w^{\lambda-1}\,g(w)\,dw_1\cdots dw_n. \tag{3.28}$$

3.3. **Laplace's method: two simple saddle points.** Once again we consider the integral (3.10). This time howerver, we suppose that $p(x)$ has two simple saddle points $x_\pm$, which means

$$p'(x_\pm) = 0 \qquad \text{and} \qquad p_\pm := p''(x_\pm) \neq 0.$$

This case is quite different from the one-dimensional case because of the Vandermonde determinant. Our assumptions are:

(i) $p(x)$ and $q(t)$ are single-valued and holomorphic in the domains $\mathbf{T}\subseteq\mathbb{C}$ and $\mathbf{T}^n$; $h(x)$ is a homogeneous analytic function of degree $\nu\geq 0$ in $\mathbb{C}$; for simplicity, we also suppose that $q(t)$ is symmetric and $h(x)=h(-x)$.

(ii) The integration path $\mathscr{P}$ is independent of N. The endpoints a and b of $\mathscr{P}$ are finite or infinite, and $(a,b)_{\mathscr{P}}$ lies within $\mathbf{T}$.

(iii) $p'(x)$ has exactly two simple zeros at interior points x_+, x_- of $\mathscr{P}$; setting $t_{j,+} = (x_+, \ldots, x_+, x_-, \ldots, x_-)$ consisting of j x_+'s and $(n-j)$ x_-'s, $p_\pm = p''(x_\pm)$ and $q_{j,+} = q(t_{j,+})$.

(iv) There exists $N_0 > 0$ such that I_{N_0} converges absolutely at $(a, \ldots, a)$ and $(b, \ldots, b)$.

(v) $\mathrm{Re}\{p(x) - p(x_\mp)\}$ is positive on $(a, x_0]_{\mathscr{P}}$ and $[x_0, b)_{\mathscr{P}}$ respectively for some $x_0 \in (a, b)_{\mathscr{P}}$, except at x_-, x_+, and is bounded away from zero as $x \to a$, or $x \to b$ along $\mathscr{P}$.

As in step 1 of Section 3.2 let $L_{1,+} \cup L_{2,+}$ and $L_{1,-} \cup L_{2,-}$ denote the unions of two half-lines respectively corresponding to x_+ and x_-. Thus, the n-dimensional integral can now be broken into a sum of 2^n terms, each being an n-dimensional integral in which each variable is in the neighborhood of either x_+ or x_-. When j variables are in the neighborhood of x_+, the Vandermonde determinant becomes

$$\Delta_{j,+}(w) = (x_+ - x_-)^{j(n-j)} \prod_{1 \le p < q \le j} (w_p - w_q) \prod_{j < p < q \le n} (w_p - w_q).$$

By the symmetry assumption, there are $\binom{n}{j}$ terms amongst the 2^n terms that produce such a Vandermonde determinant. By proceeding as for Theorem 3.5 and focusing only on the dominant contribution we rapidly establish the following theorem which contains integrals similar to (3.18):

$$B_{j,+} = \int_{(L_{1,+} \cup L_{2,+})^j \times (L_{1,-} \cup L_{2,-})^{n-j}} \exp\Big\{ - \frac{p_+}{2} \sum_{l=1}^{j} w_l^2 - \frac{p_-}{2} \sum_{l=j+1}^{n} w_l^2 \Big\} h(\Delta_{j,+}(w))\, dw_1 \cdots dw_n. \tag{3.29}$$

Theorem 3.7. *With the foregoing assumptions (i)–(v),*

$$\int_a^b \cdots \int_a^b \exp\{-N \sum_{j=1}^{n} p(t_j)\} h(\Delta(t))\, q(t)\, dt_1 \cdots dt_n =$$
$$\sum_{l=0}^{n} \binom{n}{l} \exp\{-N\big(lp(x_+) + (n-l)p(x_-)\big)\} \Big(q_{l,+}\, B_{l,+} N^{-(l_\nu + (n-l)_\nu + n)/2} \big(1 + O(N^{-1/2})\big)\Big) \tag{3.30}$$

as $N \to \infty$. Here the coefficients $B_{l,+}$ are given by (3.29).

Notice the fact that

$$l_\nu + (n-l)_\nu = \nu(l - \frac{n}{2})^2 + \nu n(n-2)/4 \quad (l = 0, 1, \ldots, n)$$

attains its minimum value at $l = m$ if $n = 2m$ or $l = m, m-1$ if $n = 2m-1$, which shows that probably only one or two terms in the sum of (3.30) give a major contribution.

3.4. **Integrals of Selberg type.** Recall that our aim is to evaluate the integrals such as (1.11) when $N \to \infty$. In Theorems 3.5 and 3.7 $h(\Delta(t))$ must be homogeneous and analytic. The latter condition is problematic since we want to calculate integrals involving $|\Delta(t)|^\nu$. Of course, when $t \in \mathbb{R}^n$ and ν is even, then we can set $h(\Delta(t)) = (\Delta(t))^\nu$. For ν not even, it will be enough to use the following integral representation:

$$|x|^\nu = c_\nu \int_0^\infty r^{-\nu-1} H_\nu(rx) dr, \tag{3.31}$$

which holds if $x \in \mathbb{R}$, $\nu > -1$ and $\nu \neq 0, 2, 4, \ldots$. Here

$$H_\nu(r) = \sum_{j=0}^{[\nu/2]} \frac{1}{(2j)!} (-r^2)^j - \cos(r) \qquad \text{and} \qquad c_\nu = \frac{2}{\pi} \sin\Big(\frac{\pi\nu}{2}\Big) \Gamma(\nu + 1). \tag{3.32}$$

This can be computed by complex analysis method when $-1 < \nu < 0$, see Gradshteyn and Ryzhik's book [26]; otherwise, first integrate by parts and use the former result.

Let us now consider

$$I_N = \int_{\mathscr{P}} \cdots \int_{\mathscr{P}} \exp\{-N \sum_{j=1}^{n} p(t_j)\} |\Delta(t)|^\nu\, q(t)\, dt_1 \cdots dt_n, \tag{3.33}$$

where the functions p and q are as mentioned previously while $\mathscr{P}$ denotes some interval $(a,b) \subseteq \mathbb{R}$ or one circle in the complex plane. Assume that ν is not even and that the path $\mathscr{P} = (a,b) \subseteq \mathbb{R}$, then

$$I_N = c_\nu \int_{(a,b)^n} \exp\{-N \sum_{j=1}^{n} p(t_j)\}\, q(t) \left(\int_0^\infty r^{-\nu-1} H_\nu(r\Delta(t))\, dr \right) d^n t.$$

By Fubini's theorem, we rewrite

$$I_N = c_\nu \int_0^\infty r^{-\nu-1} \left(\int_{(a,b)^n} \exp\{-N \sum_{j=1}^{n} p(t_j)\}\, q(t)\, H_\nu(r\Delta(t))\, d^n t \right) dr. \tag{3.34}$$

Likewise, if ν is not even and $\mathscr{P}$ is the unity circle $\mathbb{T} := \{z : |z| = 1\}$, setting $t_j = e^{i\theta_j}$, since

$$|\Delta(t)| = |\prod_{1 \le j < k \le n} (2 \sin \frac{\theta_j - \theta_k}{2})| \quad \text{and} \quad \prod_{1 \le j < k \le n} (2 \sin \frac{\theta_j - \theta_k}{2}) = \Delta(t) \prod_{j=1}^{n} (it_j)^{-(n-1)/2},$$

we have

$$I_N = c_\nu \int_0^\infty r^{-\nu-1} \left(\int_{\mathbb{T}^n} \exp\{-N \sum_{j=1}^{n} p(t_j)\}\, q(t)\, H_\nu\big(r\Delta(t) \prod_{j=1}^{n} (it_j)^{-(n-1)/2}\big)\, d^n t \right) dr. \tag{3.35}$$

Notice that the precise form of the integral over r in (3.34) and (3.35) allows a rescaling of the variable, so the function $H_\nu(r)$ plays a similar role like a homogeneous function of degree ν, and we can apply established theorems for the interior n-dimensional integrals on the right-hand side of (3.34) and (3.35). Finally, since we are concerned on the leading term as $N \to \infty$ in this article, assuming that we can deform the line segments in (3.18) or in (3.29) to the real line, then we can once more interchange the order of integration and reconstruct the absolute value by integrating over r.

In what follows, we say that the integral I_N as above satisfies the condition (vi): after an appropriate change of variable, the factor ξ in $H_\nu(r\xi)$, depending on the variables w and the saddle points, becomes a real-valued variable (the line segments L_i, coming from the integration in the neighborhood of the saddle points, should first be deformed to the real line). All the examples considered in the article satisfy this condition. The next corollaries immediately follow from Theorem 3.5, Remark 3.6, Theorem 3.7, and the use of the integral representation (3.31). Similar results hold for the integral (3.35).

Corollary 3.8. *Under the foregoing assumptions (i)-(v) of Section 3.2 and (vi) above, let*

$$I_{N,n} = \int_{(a,b)^n} \exp\{-N \sum_{j=1}^{n} p(t_j)\} |\Delta(t)|^\nu\, q(t)\, g(N^{1/\mu}(t - t_0))\, d^n t$$

where $g(t)$ is analytic in $\mathbb{C}^n$ and $p(x)$ admits one saddle point x_0 of order $\mu - 1$. Then, as $N \to \infty$,

$$I_{N,n} \sim \frac{e^{-nNp(x_0)}}{N^{(n_\nu + n)/\mu}} A_0\, q(x_0, \ldots, x_0)$$

where

$$A_0 = \int_{\mathbb{R}^n} \exp\Big\{ - p^{(\mu)}(x_0)/\mu! \sum_{j=1}^{n} w_j^\mu \Big\}\, g(w)\, |\Delta(w)|^\nu\, d^n w.$$

Corollary 3.9. *Under the foregoing assumptions (i)-(v) of Section 3.3 and (vi) above, let*

$$I_{N,n} = \int_{(a,b)^n} \exp\{-N \sum_{j=1}^{n} p(t_j)\} |\Delta(t)|^\nu\, q(t)\, d^n t$$

where $p(x)$ admits two simple saddle points x_+, x_-, and $\mathrm{Re}\{x_+ - x_-\} \ge 0$. *Moreover, let $p_\pm = p''(x_\pm)$ and $\Gamma_{\nu,m}$ be given in* (A.5). *If* $\mathrm{Re}\{p(x_+)\} = \mathrm{Re}\{p(x_-)\}$, *then as $N \to \infty$,*

$$I_{N,2m} \sim \binom{2m}{m} (\Gamma_{\nu,m})^2 \frac{(x_+ - x_-)^{\nu m^2}}{(\sqrt{p_+ p_-})^{m + \nu m(m-1)/2}} \frac{e^{-mN(p(x_+) + p(x_-))}}{N^{m + \nu m(m-1)/2}} q(x_+^m, x_-^m)$$

while

$$I_{N,2m-1} \sim \tbinom{2m-1}{m}\Gamma_{\nu,m-1}\Gamma_{\nu,m}\frac{(x_+-x_-)^{\nu m(m-1)}}{(\sqrt{p_+p_-})^{m+\nu m(m-1)/2}}\frac{e^{-mN(p(x_+)+p(x_-))}}{N^{(2m-1+\nu(m-1)^2)/2}} \times\left(e^{Np(x_+)}(\sqrt{p_+})^{1+\nu(m-1)}q(x_+^{m-1},x_-^m)+e^{Np(x_-)}(\sqrt{p_-})^{1+\nu(m-1)}q(x_+^m,x_-^{m-1})\right).$$

We conclude this section by applying the two last corollaries to the study of the asymptotic behavior of the generalized Airy function [10]:

$$\mathrm{Ai}^{(\alpha)}(s)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ip_3(t)/3}|\Delta(t)|^{2/\alpha}\ {}_0\mathcal{F}_0^{(\alpha)}(s;it)\,dt \tag{3.36}$$

where it is assumed that the variables $s=(s_1,\dots,s_n)\in\mathbb{R}^n$. Note that the generalized Airy function obviously reduces to the classical Airy function when $n=1$. Moreover, for $\alpha=1$, the above n-dimensional integral is proportional to Kontsevich's matrix Airy function $A(S)$, where S denotes a $n\times n$ hermitian matrix with eigenvalues $s_1,\dots,s_n$ (see Section 4 in [33]). It is also worth noting that in the case where $\alpha=1$ and $s_1=\ldots=s_n=u$, then the above integral representation can be reduced to a very simple determinantal formula (see for insante Section 4 in [11] or Ref. 8 therein):

$$\mathrm{Ai}^{(2)}(u,\dots,u)=(-1)^{n(n-1)/2}n!\det\left[\frac{d^{i+j-2}}{du^{i+j-2}}\mathrm{Ai}(u)\right]_{i,j=1}^{n}. \tag{3.37}$$

Before displaying the asymptotic the generalized Airy function, we recall the following shorthand notation: when $A,B\in\mathbb{R}$, $(A+Bs)$ stands for $(A+Bs_1,\dots A+Bs_n)$.

Proposition 3.10. *Let x be a real positive variable. Then as $x\to\infty$,*

$$\mathrm{Ai}^{(\alpha)}(x+x^{-1/2}s)\sim\frac{\Gamma_{2/\alpha,n}}{(2\pi)^n2^{(n+n(n-1)/\alpha)/2}}\frac{e^{-\frac{2n}{3}x^{3/2}}e^{-p_1(s)}}{x^{(n+n(n-1)/\alpha)/4}}, \tag{3.38}$$

and for $n=2m$

$$\mathrm{Ai}^{(\alpha)}(-x+x^{-1/2}s)\sim\tbinom{2m}{m}\frac{(\Gamma_{2/\alpha,m})^2}{(2\pi)^n}(2\sqrt{x})^{-m+m(m+1)/\alpha}\,e^{-ip_1(s)}{}_1F_1^{(\alpha)}(m/\alpha;n/\alpha;2is). \tag{3.39}$$

Proof. (3.38) and (3.39) originate from integrals evaluated around one simple saddle point and two simple saddle points, respectively. For (3.38), set $N=x^{3/2}$. Simple manipulations and the use of Proposition 2.2 lead to

$$\mathrm{Ai}^{(\alpha)}(N^{2/3}+N^{-1/3}s)=\frac{N^{(n+n(n-1)/\alpha)/3}}{(2\pi)^n}\int_{\mathbb{R}^n}e^{N(ip_3(t)/3+ip_1(t))}|\Delta(t)|^{2/\alpha}{}_0\mathcal{F}_0^{(\alpha)}(s;it)dt.$$

We thus have an integral like in Corollary 3.8 with $p(t_j)=-it_j^3/3-it_j$, $q(t)={}_0\mathcal{F}_0^{(\alpha)}(s;it)$, and $g(t)=1$. The function p has two simple saddle points at $\pm i$. With $x_0=i$, we have $p_0=p''(x_0)/2=1$ which implies that the steepest descent path near x_0 would follow the horizonal line, as desired. We may thus apply Corollary 3.8 to the case $\mu=2$, $x_0=i$, $p_0=1$, $p(x_0)=2/3$, $q(x_0,\dots,x_0)=e^{-p_1(s)}$ and (3.38) follows immediately.

For (3.39), we also let $N=x^{3/2}$. In the definition of $\mathrm{Ai}^{(\alpha)}(s)$, substitute t_j by $N^{1/3}t_j$ and apply Proposition 2.2, which yields

$$\mathrm{Ai}^{(\alpha)}(-N^{2/3}+N^{-1/3}s)=\frac{N^{(n+n(n-1)/\alpha)/3}}{(2\pi)^n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}e^{-N\,p(t_j)}|\Delta(t)|^{2/\alpha}{}_0\mathcal{F}_0^{(\alpha)}(is;t)dt$$

where $p(t_j)=i\left(-t_j^3/3+t_j\right)$. This function has 2 simple saddle points, namely $x_\pm=\pm1$. This time we have to consider both of them because they are already on the path of integration. We have $p(x_\pm)=\pm2i/3$, $p_\pm=p''(x_\pm)=\mp2i$. This means that the steepest descent path is given by

$$\mathscr{P}=\left\{-1+\tau e^{-i\pi/4}\,:\,\tau\in(-\infty,\sqrt{2}]\right\}\cup\left\{1+\tau e^{i\pi/4}\,:\,\tau\in[-\sqrt{2},\infty)\right\}.$$

By making the change of variables $w_j \mapsto e^{\pm i\pi/4} v_j$ (for saddle points $x_\pm$) in (3.29), we see that the variables v_j follow the real line, so the assumption (vi) is fulfilled and (3.39) follows from the previous corollary. □

4. Scaling limits

In this section, we prove the theorems and corollaries given in the introduction. Most of the proofs rely on Corollaries 3.8 and 3.9 or similar results for integrals on the torus $\mathbb{T}^n$, more precisely for (3.35). All the integrals considered here fulfill the assumptions (i) to (vi) given in the last section. Note that when we talk about deforming the contours of integration, we implicitly suppose that either the power ν of the absolute value of Vandermonde determinant is even and the variables are real, or the integral representation (3.31) is being used. Also in this section is the assumption that a deformation of contours is made as long as the integrand is analytic and absolutely integrable over the concerned region of the complex plane.

We will frequently use the symbol $\mathscr{L}_a^b$ to denote a straight line path from a to b. Additionally, $\mathscr{M}_a^b$ will denote a semi-circular path in the positive direction, starting at a and ending at b with radius $|a-b|/2$. Dyson index β and its duality $\beta' := 4/\beta$ will be used alternately.

Before starting our computation we first review the integral representation of hypergeometric functions ${}_2F_1^{(\alpha)}(a,b;c;s)$ and ${}_1F_1^{(\alpha)}(a;c;s)$, due to Yan [51] and Forrester [21], especially [23] for more details. For $\alpha > 0$, $\mathrm{Re}\{\nu_1\} > -1$ and $\mathrm{Re}\{\nu_2\} > -1$,

$$ {}_2F_1^{(\alpha)}(a,b;c;s_1,\ldots,s_n) = \frac{1}{S_n(\nu_1,\nu_2,1/\alpha)} \int_{[0,1]^n} {}_1\mathcal{F}_0^{(\alpha)}(a;s;t)\, D_{\nu_1,\nu_2,1/\alpha}(t)\, d^n t, \tag{4.1} $$

where $\nu_1 = b-(n-1)/\alpha - 1$, $\nu_2 = c-b-(n-1)/\alpha-1$ and

$$ D_{\nu_1,\nu_2,1/\alpha}(t) = \prod_{i=1}^n t_i^{\nu_1}(1-t_i)^{\nu_2} \prod_{1\le i<j\le n} |t_i-t_j|^{2/\alpha}. \tag{4.2} $$

If $\mathrm{Re}\{\nu_1\} > -1$, the right-hand integral of (4.1) can be analytically continued so that it is valid for $\mathrm{Re}\{\nu_2\} \le -1$ but by replacing the interval $[0,1]$ with the counter-clockwise circle $\mathbb{T}$, especially in the case of interest $(a=-N)$ [4, 21]:

$$ \begin{aligned} {}_2F_1^{(\alpha)}(a,b;c;s) = {} & \frac{e^{i\pi(b-c)n}}{M_n(b-c,c+1+(n-1)/\alpha,1/\alpha)} \\ & \times \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} {}_1\mathcal{F}_0^{(\alpha)}(a;s;1-t)\, D_{\nu_2+(n-1)/\alpha,\nu_1,1/\alpha}(t)\, d^n t. \end{aligned} \tag{4.3} $$

Note that the constant $M_n(a,b,\alpha)$ is given in the appendix. Likewise, for $\mathrm{Re}\{c-a\} > -1$, we have

$$ {}_1F_1^{(\alpha)}(a;c+1+(n-1)/\alpha;s) = \frac{e^{-i\pi n a}}{M_n(-a,c,1/\alpha)} \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \prod_{j=1}^n t_j^{a-1}(1-t_j)^{c-a} |\Delta(t)|^{2/\alpha}\, {}_0\mathcal{F}_0^{(\alpha)}(s;t)\, d^n t. \tag{4.4} $$

4.1. **Hermite β-ensemble.** It follows from the duality relation in Proposition 7 [10] that

$$ \begin{aligned} K_N(s_1,\ldots,s_n) : &= \Big\langle \prod_{i=1}^N \prod_{j=1}^n (x_i - s_j) \Big\rangle_{\mathrm{H}\beta\mathrm{E}} \\ &= (-i)^{nN} 2^{\beta' n(n-1)/4+n/2} (\Gamma_{\beta',n})^{-1} e^{p_2(s)} \int_{\mathbb{R}^n} \prod_{j=1}^n t_j^N e^{-t_j^2} |\Delta(t)|^{\beta'}\, {}_0\mathcal{F}_0^{(2/\beta')}(-2is;t) d^n t. \end{aligned} $$

Set

$$ s_j \mapsto \sqrt{2N}\left(u + \frac{s_j}{\rho N}\right) \qquad \text{and} \qquad t_j \mapsto \sqrt{2N} t_j. $$

By using Proposition 2.2, the weighted quantity

$$ \varphi_N(s_1,\ldots,s_n) := e^{-\frac{1}{2}p_2(s)} K_N(s_1,\ldots,s_n) $$

becomes

$$\varphi_{N-l}\big(\sqrt{2N}(u+\frac{s}{\rho N})\big) = (-i\sqrt{2N})^{n(N-l)}(2\sqrt{N})^{\beta' n(n-1)/2+n}(\Gamma_{\beta',n})^{-1}e^{nNu^2+p_2(s)/(\rho^2 N)}I_{N,n}, \tag{4.5}$$

where

$$\begin{aligned} I_{N,n} &= \int_{\mathbb{R}^n} \exp\{-N\sum_{j=1}^n p(t_j)\}|\Delta(t)|^{\beta'}q(t)d^n t \\ &= c_{\beta'}\int_0^\infty r^{-\beta'-1}\bigg(\int_{\mathbb{R}^n} \exp\{-N\sum_{j=1}^n p(t_j)\}\,q(t)\,H_{\beta'}(r\Delta(t))\,d^n t\bigg)dr, \end{aligned} \tag{4.6}$$

and

$$p(x) = 2x^2+4iux-\ln x, \qquad q(t) = \prod_{j=1}^n t_j^{-l}\,{}_0\mathcal{F}_0^{(2/\beta')}(-4is/\rho; t+iu/2). \tag{4.7}$$

Here l is a fixed integer, and it is assumed that $l=0$ for $n=2m$ while $l=0,1$ for $n=2m-1$.

Since $p'(x) = 4x+4iu-\frac{1}{x}$, there are two simple saddle points in the bulk: $x_\pm = \frac{-iu\pm\sqrt{1-u^2}}{2}$ where $u\in(-1,1)$. At the rightmost soft edge (which corresponds to $u=1$), these two points coincide and give a single saddle point $x_0 = -\frac{i}{2}$ of multiplicity 2.

In the following lines, we proceed to compute the leading asymptotic terms in the bulk and at the soft edge. We start with the bulk of the spectrum. We set $u=\sin\theta$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, so that $x_+ = \frac{1}{2}e^{-i\theta}$, $x_- = \frac{1}{2}e^{i(\theta+\pi)}$ and

$$p(x_+) = -\frac{1}{2}\cos 2\theta + (1+\ln 2) + i(\theta+\frac{1}{2}\sin 2\theta), \quad p(x_-) = -\frac{1}{2}\cos 2\theta + (1+\ln 2) - i(\theta+\frac{1}{2}\sin 2\theta+\pi).$$

It follows from

$$p_\pm := p''(x_\pm) = 8e^{\pm i\theta}\cos\theta$$

that the angles of steepest descent are $-\theta/2$ at x_+ and $\theta/2$ at x_-. Note that if $x = -\frac{i}{2}u+v$, $v\in\mathbb{R}$, then the function

$$\mathrm{Re}\{p(x)\} = 2v^2 - \frac{1}{2}\ln(v^2+\frac{1}{4}u^2)+\frac{3}{2}u^2$$

attains its minimum value at the point $v=\pm\frac{1}{2}\sqrt{1-u^2}$. Therefore, as a possible path, we consider the straight line passing through $-\frac{i}{2}u$ and parallel to the real axis (together with irrelevant deformations at $\pm\infty$).

The segments of integration $(-\infty e^{-i\frac{\theta}{2}},\infty e^{-i\frac{\theta}{2}})$ and $(-\infty e^{i\frac{\theta}{2}},\infty e^{i\frac{\theta}{2}})$ can be deformed into the real axis near the saddle points. Since $x_+-x_->0$, the assumption (vi) is fulfilled. Take $\rho=\frac{2}{\pi}\sqrt{1-u^2}$. According to Corollary 3.9, for $n=2m$ $(l=0)$,

$$\begin{aligned} I_{N,2m} &\sim (8N)^{-\beta' m(m-1)/2-m}\exp\{-nNu^2-nN(1+2\ln 2-i\pi)/2\}\,(\frac{\pi\rho}{2})^{\beta' m(m+1)/2-m} \\ &\quad\times \tbinom{2m}{m}(\Gamma_{\beta',m})^2\,{}_0\mathcal{F}_0^{(2/\beta')}(i\pi s;1^m,(-1)^m), \end{aligned} \tag{4.8}$$

while for $n=2m-1$,

$$\begin{aligned} I_{N,2m-1} &\sim (8N)^{-\beta'(m-1)^2/2-n/2}\exp\{-nNu^2-nN(1+2\ln 2-i\pi)/2\}\,(\frac{\pi\rho}{2})^{\beta'(m^2-1)/2-n/2} \\ &\quad\times \tbinom{2m-1}{m}\Gamma_{\beta',m-1}\Gamma_{\beta',m}\,(-2i)^{(2m-1)l}\Big(e^{i\theta_N-il(\theta+\frac{\pi}{2})}E_{m-1}^{(2/\beta')}(i\pi s)+e^{-i\theta_N+il(\theta+\frac{\pi}{2})}E_{m-1}^{(2/\beta')}(-i\pi s)\Big), \end{aligned} \tag{4.9}$$

where

$$\theta_N = N(2\theta+\sin 2\theta+\pi)/2+\theta(1+(m-1)\beta')/2, \qquad \theta=\arcsin u.$$

Hence, for $n=2m$,

$$\varphi_N\big(\sqrt{2N}(u+\frac{s}{\rho N})\big) \sim \Psi_{N,2m}\gamma_m(\beta')\,{}_0\mathcal{F}_0^{(2/\beta')}(i\pi s;1^{(m)},(-1)^{(m)}), \tag{4.10}$$

where

$$\Psi_{N,2m} = (\pi\rho)^{\beta' m(m+1)/2-m}N^{\beta' m^2/2}\exp\{-mN(1+\ln 2-\ln N)\}$$

and

$$\gamma_m(\beta') := \binom{2m}{m} \prod_{j=1}^{m} \frac{\Gamma(1+\beta' j/2)}{\Gamma(1+\beta'(m+j)/2)}. \tag{4.11}$$

For $n = 2m-1$,

$$\varphi_{N-l}\big(\sqrt{2N}(u+\frac{s}{\rho N})\big) \sim \Psi^{(l)}_{N,n} \frac{1}{2i\sqrt{\cos\theta}} \Big(e^{i\theta_N - il(\theta+\frac{\pi}{2})} E^{(2/\beta')}_{m-1}(i\pi s) + e^{-i\theta_N + il(\theta+\frac{\pi}{2})} E^{(2/\beta')}_{m-1}(-i\pi s) \Big), \tag{4.12}$$

where

$$\Psi^{(l)}_{N,2m-1} = \tbinom{2m-1}{m} \Gamma_{\beta',m-1}\Gamma_{\beta',m}(\Gamma_{\beta',2m-1})^{-1} (\pi\rho)^{\beta'(m^2-1)/2-(2m-1)/2} N^{\beta' m(m-1)/2}$$
$$\times \exp\{-(2m-1)N(1+\ln 2-\ln N)/2\}\,(\sqrt{N/2})^{-(2m-1)l}(2i\sqrt{\pi\rho/2}).$$

We now turn attention to the soft edge of the spectrum (i.e., $u=1$). In Eqs.(4.5)–(4.7), let $u=1$, $l=0$ and $\rho = 2N^{-1/3}$. Then,

$$p(x) = 2x^2 + 4ix - \ln x, \qquad q(t) = {}_0\mathcal{F}^{(2/\beta')}_0(-2iN^{1/3}s; t+i/2).$$

At the double saddle point $x_0 = -i/2$, we have

$$p(x_0) = 3/2 + \ln 2 + i\pi/2, \qquad p'''(x_0) = 16i.$$

The angle of steepest descent are thus $-5\pi/6$ and $-\pi/6$. However, we see that $\mathrm{Re}\{p(x+x_0)-p(x_0)\} > 0$ for all $x \in \mathbb{R}$ except at the origin. We thus choose $\mathscr{P}$ to follow the real line from $-\infty$ to ∞. Corollary 3.8 then implies

$$I_{N,n} \sim (8N)^{-\beta' n(n-1)/6 - n/3} \exp\{-nN(3+2\ln 2+i\pi)/2\}\,(2\pi)^n \mathrm{Ai}^{(2/\beta')}(s). \tag{4.13}$$

Thus,

$$\varphi_N\big(\sqrt{2N}(1+\frac{s}{\rho N})\big) \sim \Phi_{N,n}\,(2\pi)^n (\Gamma_{\beta',n})^{-1} \mathrm{Ai}^{(2/\beta')}(s), \tag{4.14}$$

where

$$\Phi_{N,n} = N^{\beta' n(n-1)/12 + n/6} \exp\{-nN(1+\ln 2-\ln N - 2i\pi)/2\}.$$

For β even, the scaling limits of the correlation functions for the HβE immediately follow from (4.10) and (4.14). Let $n = 2m = k\beta$, we have the following relation

$$R_{k,N}(x_1,\dots,x_k) = \frac{(k+N)!}{N!} \frac{G_{\beta,N}}{G_{\beta,k+N}} \prod_{1\le j<l\le k} (x_j - x_l)^\beta \left[\varphi_N(s_1,\dots,s_n)\right]_{\{s\}\mapsto\{x\}}. \tag{4.15}$$

One easily shows that as $N\to\infty$,

$$\frac{G_{\beta,N}}{G_{\beta,k+N}} \sim (2\pi)^{-k/2} 2^{\beta k(k+1)/4} \beta^{-\beta k/2} (\Gamma(1+\beta/2))^k (2e)^{\beta kN/2} N^{-\beta kN/2 - \beta k(k+1)/4 - k/2}.$$

We consider the bulk scaling and let $\rho = \frac{2}{\pi}\sqrt{1-u^2}$. Some manipulations then lead to

$$\big(\frac{\sqrt{2N}}{\rho N}\big)^k R_{k,N}\big(\sqrt{2N}(u+\frac{x}{\rho N})\big) \sim (\beta/2)^{-\beta k/2} (\Gamma(1+\beta/2))^k \gamma_m(\beta')\,|\Delta(2\pi x)|^\beta\,{}_0\mathcal{F}^{(2/\beta')}_0(i\pi s; 1^m, (-1)^m).$$

According to Gauss's multiplication formulas for the gamma function [2],

$$\prod_{j=1}^{l} \Gamma\big(a + \frac{j-1}{l}\big) = l^{-la+\frac12}(2\pi)^{\frac{l-1}{2}}\Gamma(la), \qquad l \in \mathbb{N},$$

we have for $l = \beta/2$,

$$\gamma_m(\beta') := \binom{2m}{m} \prod_{j=1}^{m} \frac{\Gamma(1+\beta' j/2)}{\Gamma(1+\beta'(m+j)/2)} = (\beta/2)^{\beta k^2/2} \prod_{j=0}^{k-1} \frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta(k+j)/2)}\,.$$

This further implies

$$\big(\frac{\sqrt{2N}}{\rho N}\big)^k R_{k,N}\big(\sqrt{2N}(u+\frac{x}{\rho N})\big) \sim b_k(\beta)\,|\Delta(2\pi x)|^{\beta}\,{}_0\mathcal{F}_0^{(\beta/2)}(i\pi s; 1^m, (-1)^m)_{\{s\}\mapsto\{x\}}\,. \tag{4.16}$$

Note that the coefficient

$$b_k(\beta) := (\beta/2)^{\beta k(k-1)/2}(\Gamma(1+\beta/2))^k \prod_{j=0}^{k-1}\frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta(k+j)/2)} \tag{4.17}$$

is exactly same as that in the circular β-ensemble (Proposition 13.2.3, [23]) as well as in the LβE and the JβE below. This strongly suggests the universality of $b_k(\beta)$.

Similarly, for the soft edge, one sets $\rho = 2N^{-1/3}$ and gets

$$\big(\frac{\sqrt{2N}}{\rho N}\big)^k R_{k,N}\big(\sqrt{2N}(1+\frac{x}{\rho N})\big) \sim a_k(\beta)\,|\Delta(x)|^{\beta}\,\mathrm{Ai}^{(\beta/2)}(s)_{\{s\}\mapsto\{x\}}, \tag{4.18}$$

where

$$a_k(\beta) := (\beta/2)^{(\beta k+1)k}(\Gamma(1+\beta/2))^k \prod_{j=1}^{2k}\frac{(\Gamma(1+2/\beta))^{\beta/2}}{\Gamma(1+\beta j/2)}. \tag{4.19}$$

4.2. **Laguerre β-ensemble.** Based on the work of Kaneko [28], it is easy to verify that if

$$K_N(s_1,\ldots,s_n) = \Big\langle \prod_{i=1}^{N}\prod_{j=1}^{n}(x_i - s_j)\Big\rangle_{\mathrm{L}\beta\mathrm{E}},$$

then

$$K_N(s_1,\ldots,s_n) = \frac{W_{\lambda_1+n,\beta,N}}{W_{\lambda_1,\beta,N}}\,{}_1F_1^{(\beta/2)}\left(-N; (2/\beta)(\lambda_1+n); s_1,\ldots,s_n\right), \tag{4.20}$$

for instance, see Proposition 13.2.5, [23]. By making the change $s_i \mapsto s_i/N$ and taking the limit $N\to\infty$, one readily proves the first formula of Theorem 1.3. Note that this result could be obtained by taking the integral duality formula in [10] and following the asymptotic method developed in the previous section for the case where $p(x)$ admits a simple saddle point.

By using (4.4), we get the following integral formula

$$K_N(s_1,\ldots,s_n) = A_N\, i^{-n}\int_{\mathbb{T}^n}\prod_{j=1}^{n} t_j^{-N-1}(1-t_j)^{N-1+\beta'(\lambda_1+1)/2}|\Delta(t)|^{\beta'}\,{}_0\mathcal{F}_0^{(2/\beta')}(s;t)\,d^n t \tag{4.21}$$

where

$$A_N = \frac{(2\pi)^{-n}e^{i\pi nN}}{M_n(N,-1+\beta'(\lambda_1+1)/2,\beta'/2)}\frac{W_{\lambda_1+n,\beta,N}}{W_{\lambda_1,\beta,N}}. \tag{4.22}$$

Now let

$$\varphi_N(s_1,\ldots,s_n) := e^{-\frac{1}{2}p_1(s)}\prod_{1\le j\le n} s_j^{\lambda_1/\beta}\,K_N(s_1,\ldots,s_n).$$

The application of Proposition 2.2 then yields

$$\varphi_{N-l}\big(4N(u+\frac{s}{\rho N})\big) = A_{N-l}\,(4N)^{n\lambda_1/\beta}e^{-2nNu}\prod_{1\le j\le n}\big(u+\frac{s_j}{\rho N}\big)^{\lambda_1/\beta} I_{N,n}, \tag{4.23}$$

where

$$\begin{aligned} I_{N,n} &= \int_{\mathbb{T}^n}\exp\{-N\sum_{j=1}^{n}p(t_j)\}|\Delta(t)|^{\beta'}q(t)d^n t \\ &= c_{\beta'}\int_0^\infty r^{-\beta'-1}\Big(\int_{\mathbb{T}^n}\exp\{-N\sum_{j=1}^{n}p(t_j)\}\,q(t)\,H_{\beta'}\big(r\Delta(t)\prod_{j=1}^{n}(it_j)^{-(n-1)/2}\big)\,d^n t\Big)dr, \end{aligned} \tag{4.24}$$

and

$$p(x) = \ln x - \ln(1-x) - 4ux, \qquad q(t) = i^{-n} \prod_{j=1}^{n} t_j^{l-1}(1-t_j)^{-l-1+\beta'(\lambda_1+1)/2} \, {}_0\mathcal{F}_0^{(2/\beta')}(4s/\rho; t-1/2). \quad (4.25)$$

Since $p'(x) = \frac{1}{x(1-x)} - 4u$, there are two simple saddle points in the bulk, namely $x_\pm = \frac{1}{2} \pm \frac{i}{2}\sqrt{\frac{1-u}{u}}$ with $u \in (0,1)$. By letting $u \to 1$, which corresponds to the soft edge, we find that the two saddle points become one double saddle point $x_0 = \frac{1}{2}$.

We first focus on the bulk of the spectrum. Set $u = \cos^2\theta,\ \theta \in (0, \frac{\pi}{2})$, hence $x_\pm = \frac{1}{2\cos\theta} e^{\pm i\theta}$, and

$$p(x_\pm) = -2\cos^2\theta \pm i(2\theta - \sin 2\theta), \qquad p_\pm := p''(x_\pm) = \pm 16iu^2\sqrt{\tfrac{1-u}{u}}.$$

We see that the angles of steepest descent are $\frac{3}{4}\pi$ at x_+ and $\frac{1}{4}\pi$ at x_-, so we choose the following path of integration:

$$\mathscr{L}_1^{1/(2\cos\theta)} \bigcup \mathscr{M}_{1/(2\cos\theta)}^{-1/(2\cos\theta)} \bigcup \mathscr{M}_{-1/(2\cos\theta)}^{1/(2\cos\theta)} \bigcup \mathscr{L}_{1/(2\cos\theta)}^{1}.$$

Actually, when $x \in \mathscr{L}_1^{1/(2\cos\theta)}$ or $\mathscr{L}_{1/(2\cos\theta)}^{1}$, we have

$$\mathrm{Re}\{p(x) - p(x_+)\} = \ln \frac{x}{|1-x|} - 4ux + 2u > 0.$$

On the semicircle $\mathscr{M}_{1/(2\cos\theta)}^{-1/(2\cos\theta)}$, we can write $x = \frac{1}{2\cos\theta} e^{i\phi}$ with $\phi \in [0, \pi]$. Since

$$g(\cos\phi) := \mathrm{Re}\{p(x) - p(x_+)\} = -\tfrac{1}{2}\ln(1 + 4\cos^2\theta - 4\cos\theta\cos\phi) + 2(\cos\theta - \cos\phi)\cos\theta,$$

it follows from

$$g'(\cos\phi) = \frac{8(\cos\phi - \cos\theta)\cos^2\theta}{1 + 4\cos^2\theta - 4\cos\theta\cos\phi}$$

that $g(\cos\phi)$ attains its minimum value at $\cos\phi = \cos\theta$. Similar results hold for $\mathrm{Re}\{p(x) - p(x_-)\}$.

Case 1: $n = 2m$.

The line segments of integration $(-\infty e^{i\frac{3}{4}\pi}, \infty e^{i\frac{3}{4}\pi})$ and $(-\infty e^{i\frac{1}{4}\pi}, \infty e^{i\frac{1}{4}\pi})$ become the real line by making the following change of variables: $w_j \mapsto w_j e^{i\frac{3}{4}\pi}$ and $w_j \mapsto w_j e^{i\frac{1}{4}\pi}$, respectively near x_+ and x_-. We take $\rho = \frac{2}{\pi}\sqrt{\frac{1-u}{u}}$ and $l = 0$. Then, according to Theorem 3.7, we have

$$\begin{aligned} I_{N,2m} &\sim N^{-\beta' m(m-1)/2 - m} e^{4mNu} (2\sqrt{u})^{-\beta' m\lambda_1} \left(\frac{\pi\rho}{2}\right)^{\beta' m(m+1)/2 - m} \\ &\quad \times \binom{2m}{m} (\Gamma_{\beta',m})^2 \, {}_0\mathcal{F}_0^{(2/\beta')}(i\pi s; 1^m, (-1)^m). \end{aligned} \quad (4.26)$$

¿From

$$\frac{W_{\lambda_1+n,\beta,N}}{W_{\lambda_1,\beta,N}} = (2/\beta)^{nN} \prod_{j=0}^{N-1} \frac{\Gamma(1+\lambda_1+n+\beta j/2)}{\Gamma(1+\lambda_1+\beta j/2)} = \prod_{j=1}^{n} \frac{\Gamma(N+\beta'(\lambda_1+j)/2)}{\Gamma(\beta'(\lambda_1+j)/2)},$$

we also get

$$\begin{aligned} A_N &= (2\pi)^{-n} e^{i\pi nN} \prod_{j=1}^{n} \frac{\Gamma(1+\beta'/2)\Gamma(1+N+\beta'(n-j)/2)}{\Gamma(1+\beta' j/2)} \\ &\sim (\Gamma_{\beta',n})^{-1} N^{\beta' n(n-1)/4 + n/2} \exp\{-nN(1 - \ln N - i\pi)\}. \end{aligned} \quad (4.27)$$

Hence,

$$\varphi_N\big(4N(u + \frac{s}{\rho N})\big) \sim \Psi_{N,2m} \gamma_m(\beta') \, {}_0\mathcal{F}_0^{(2/\beta')}(i\pi s; 1^m, (-1)^m), \quad (4.28)$$

where

$$\Psi_{N,2m} = (\pi\rho/2)^{\beta' m(m+1)/2 - m} N^{\beta' m(m+\lambda_1)/2} \exp\{-2mN(1 - \ln N)\}.$$

Case 2: $n = 2m-1$. According to Theorem 3.7, two types of integration domains lead to a significant contribution as $N \to \infty$. They correspond to the intervals of integration

$$(-\infty e^{i\frac{3}{4}\pi}, \infty e^{i\frac{3}{4}\pi})^{m-1} \times (-\infty e^{i\frac{1}{4}\pi}, \infty e^{i\frac{1}{4}\pi})^{m}, \qquad (-\infty e^{i\frac{3}{4}\pi}, \infty e^{i\frac{3}{4}\pi})^{m} \times (-\infty e^{i\frac{1}{4}\pi}, \infty e^{i\frac{1}{4}\pi})^{m-1}$$

which can be deformed to (whenever $m > 1$)

$$(-\infty e^{i\frac{3}{4}\pi}, \infty e^{i\frac{3}{4}\pi})^{m-1} \times (-\infty e^{\frac{i}{4}\pi + \frac{i}{m}(\frac{\pi}{2}-2\theta)}, \infty e^{\frac{i}{4}\pi + \frac{i}{m}(\frac{\pi}{2}-2\theta)})^{m}$$

and

$$(-\infty e^{i\frac{3}{4}\pi + \frac{i}{m}(2\theta - \frac{\pi}{2})}, \infty e^{i\frac{3}{4}\pi + \frac{i}{m}(2\theta-\frac{\pi}{2})})^{m} \times (-\infty e^{i\frac{1}{4}\pi}, \infty e^{i\frac{1}{4}\pi})^{m-1}.$$

Then, by changing the variables as in $n = 2m$ case, we get

$$\begin{aligned} I_{N,2m-1} &\sim N^{-\beta'(m-1)^2/2 - n/2} \exp\{2nNu\} \, (\frac{\pi\rho}{2})^{\beta'(m^2-1)/2 - n/2} (2\sqrt{u})^{-\beta'(1+n\lambda_1)/2} \\ &\times \tbinom{2m-1}{m} \Gamma_{\beta',m-1}\Gamma_{\beta',m} \frac{1}{i}\Big(e^{i(\theta_N - 2l\theta)} E^{(2/\beta')}_{m-1}(-i\pi s) - e^{-i(\theta_N - 2l\theta)} E^{(2/\beta')}_{m-1}(i\pi s)\Big), \end{aligned} \tag{4.29}$$

where

$$\theta_N = N(2\theta - \sin 2\theta) + \beta'(\lambda_1+1)\theta/2 + \beta'(m-1)(\theta - \pi/4) + \pi/4, \qquad \theta = \arccos\sqrt{u}.$$

One finally obtains

$$\varphi_{N-l}\big(4N(u + \frac{s}{\rho N})\big) \sim \Psi^{(l)}_{N,n} \frac{1}{2\sqrt{2} i \sqrt[4]{u(1-u)}} \Big(e^{i(\theta_N - 2l\theta)} E^{(2/\beta')}_{m-1}(-i\pi s) - e^{-i(\theta_N - 2l\theta)} E^{(2/\beta')}_{m-1}(i\pi s)\Big), \tag{4.30}$$

where

$$\begin{aligned} \Psi^{(l)}_{N,2m-1} &= \tbinom{2m-1}{m} \Gamma_{\beta',m-1}\Gamma_{\beta',m} (\Gamma_{\beta',2m-1})^{-1} (\pi\rho/2)^{\beta'(m^2-1)/2 - m+1} \sqrt{2} (2\sqrt{u})^{-\beta'/2+1} \\ &\times N^{\beta' m(m-1)/2 + \beta'(2m-1)\lambda_1/4} \exp\{-(2m-1)N(1 - \ln N - i\pi)\} \, (-N)^{-(2m-1)l}. \end{aligned}$$

We now consider the soft edge of the spectrum ($u = 1$). In Eqs.(4.23)–(4.25), let $u = 1$, $l = 0$ and $\rho = 2(2N)^{-1/3}$. This yields

$$p(x) = \ln x - \ln(1-x) - 4x, \qquad q(t) = i^{-n} \prod_{j=1}^{n} t_j^{-1} (1-t_j)^{-1+\beta'(\lambda_1+1)/2} \, {}_0\mathcal{F}^{(2/\beta')}_0(4s/\rho; t - 1/2).$$

At the double saddle point $x_0 = 1/2$, we have

$$p(x_0) = -2, \qquad p'''(x_0) = 32.$$

Thus, the angles of steepest descent are $2\pi/3$ and $4\pi/3$. Since on the circle $x = \frac{1}{2}e^{i\phi}$, we have

$$\mathrm{Re}\{p(x + x_0) - p(x_0)\} = -\ln\sqrt{5 - 4\cos\phi} - 2\cos\phi + 2 > 0$$

whenever $\phi \in (0, 2\pi)$. We choose $\mathscr{P}$ to be the following path: it starts at 1, arrives at $1/2 + i0^+$ by following a straight line, along the centered circle of radius $1/2$ counterclockwisely, then follows a straight line from $1/2 - i0^+$ to 1. Applying Theorem 3.5 and Remark 3.6, noting that the line segments of integration near the saddle point can be chosen to be $(-\infty i, \infty i)$, after the change of variables: $w_j \mapsto 16^{-1/3} i w_j$ one obtains

$$I_{N,n} \sim 2^{-\beta' n(n-1)/6 - \beta' n(\lambda_1+1)/2 + 2n/3} N^{-\beta' n(n-1)/6 - n/3} \exp\{2nN\} \, (2\pi)^n \mathrm{Ai}^{(2/\beta')}(s). \tag{4.31}$$

Thus,

$$\varphi_N\big(4N(1 + \frac{s}{\rho N})\big) \sim \Phi_{N,n} \, (2\pi)^n (\Gamma_{\beta',n})^{-1} \mathrm{Ai}^{(2/\beta')}(s), \tag{4.32}$$

where

$$\Phi_{N,n} = 2^{-\beta' n(n-1)/6 - \beta' n/2 + 2n/3} N^{\beta' n(n-1)/12 + \beta' n\lambda_1/4 + n/6} \exp\{-nN(1 - \ln N - i\pi)\}.$$

When β is even, the scaling limit of the correlation functions for the LβE immediately follow from (4.28) and (4.32). For the bulk case, let $n = 2m = k\beta$ and $\rho = \frac{2}{\pi}\sqrt{\frac{1-u}{u}}$. By making use of

$$R_{k,N}(x_1,\ldots,x_k) = \frac{(k+N)!}{N!}\frac{W_{\lambda_1,\beta,N}}{W_{\lambda_1,\beta,k+N}}\prod_{1\le j<l\le k}(x_j-x_l)^\beta\left[\varphi_N(s_1,\ldots,s_n)\right]_{\{s\}\mapsto\{x\}}. \tag{4.33}$$

and

$$\frac{W_{\lambda_1,\beta,N}}{W_{\lambda_1,\beta,k+N}} \sim (2\pi)^{-k}(\beta/2)^{-\beta k/2}(\Gamma(1+\beta/2))^k e^{\beta kN} N^{-\beta kN-\beta k^2/2-(\lambda_1+1)k},$$

one can show that

$$\big(\frac{4}{\rho}\big)^k R_{k,N}\big(4N(u+\frac{x}{\rho N})\big) \sim b_k(\beta)\,|\Delta(2\pi x)|^\beta\,{}_0\mathcal{F}_0^{(\beta/2)}(i\pi s;1^m,(-1)^m)_{\{s\}\mapsto\{x\}}. \tag{4.34}$$

For the soft edge, one lets $\rho = 2(2N)^{-1/3}$ and gets

$$\big(\frac{4}{\rho}\big)^k R_{k,N}\big(4N(1+\frac{x}{\rho N})\big) \sim a_k(\beta)\,|\Delta(x)|^\beta\,\mathrm{Ai}^{(\beta/2)}(s)_{\{s\}\mapsto\{x\}}. \tag{4.35}$$

Notice that the coefficients $a_k(\beta)$ and $b_k(\beta)$ are the same as those in the HβE.

4.3. **Jacobi β-ensemble.** The JβE case is very similar to the LβE. First, Kaneko [28] proved that

$$\begin{aligned} K_N(s_1,\ldots,s_n) &:= \Big\langle\prod_{i=1}^N\prod_{j=1}^n(x_i-s_j)\Big\rangle_{\mathrm{J}\beta\mathrm{E}} \\ &= \frac{S_N(\lambda_1+n,\lambda_2,\beta/2)}{S_N(\lambda_1,\lambda_2,\beta/2)}\,{}_2F_1^{(\beta/2)}\left(-N,(2/\beta)(\lambda_1+\lambda_2+n+1)+N-1;(2/\beta)(\lambda_1+n);s\right). \end{aligned} \tag{4.36}$$

By making the change $s_i \mapsto s_i/N^2$ and taking the limit $N\to\infty$, one readily proves the second formula of Theorem 1.3.

By using (4.3) we get the following integral formula

$$K_N(s) = B_N\, i^{-n}\int_{\mathbb{T}^n}\prod_{j=1}^n t_j^{-\beta'(\lambda_2+1)/2-N}(1-t_j)^{N-2+\beta'(\lambda_1+\lambda_2+2)/2}|\Delta(t)|^{\beta'}\,{}_1\mathcal{F}_0^{(2/\beta')}(-N;s;1-t)\,d^nt \tag{4.37}$$

where

$$B_N = \frac{(2\pi)^{-n}e^{i\pi(\beta' n(\lambda_2+1)/2+n(N-1))}}{M_n(\beta'(\lambda_2+1)/2+N-1,-1+\beta'(\lambda_1+1)/2,\beta'/2)}\frac{S_N(\lambda_1+n,\lambda_2,\beta/2)}{S_N(\lambda_1,\lambda_2,\beta/2)}. \tag{4.38}$$

For the weighted quantity

$$\varphi_N(s_1,\ldots,s_n) := \prod_{1\le j\le n} s_j^{\lambda_1/\beta}(1-s_j)^{\lambda_2/\beta}\,K_N(s_1,\ldots,s_n),$$

application of Proposition 2.2 gives

$$\varphi_{N-l}\big(u+\frac{s}{\rho N}\big) = B_{N-l}\prod_{1\le j\le n}\big(u+\frac{s_j}{\rho N}\big)^{\lambda_1/\beta}\big(1-u-\frac{s_j}{\rho N}\big)^{\lambda_2/\beta}\,I_{N,n}, \tag{4.39}$$

where

$$I_{N,n} = c_{\beta'}\int_0^\infty r^{-\beta'-1}\bigg(\int_{\mathbb{T}^n}\exp\{-N\sum_{j=1}^n p(t_j)\}\,q(t)\,H_{\beta'}\big(r\Delta(t)\prod_{j=1}^n(it_j)^{-(n-1)/2}\big)\,d^nt\bigg)dr, \tag{4.40}$$

and $p(x) = \ln x - \ln(1-x) - \ln(1-u+ux)$,

$$q(t) = i^{-n}\prod_{j=1}^n t_j^{-\beta'(\lambda_2+1)/2+l}(1-t_j)^{\beta'(\lambda_1+\lambda_2+2)/2-l-2}\,{}_1\mathcal{F}_0^{(2/\beta')}(-N;\frac{s}{\rho N};\frac{1-t}{1-u+ut}). \tag{4.41}$$

Since

$$p'(x) = \tfrac{1}{x(1-x)} - \frac{u}{1-u+ux},$$

there are two simple saddle points $x_+ = \sqrt{(1-u)/u}\, e^{i\pi/2}$ and $x_- = \sqrt{(1-u)/u}\, e^{3i\pi/2}$ in the bulk with $u \in (0,1)$.

Set $u = \cos^2\theta,\ \theta \in (0, \pi/2)$, then

$$p(x_\pm) = \pm 2i\theta, \qquad p_\pm := p''(x_\pm) = 2u^2/\sqrt{u(1-u)}\ e^{\pm i(2\theta - \pi/2)}.$$

This allows us to take the angles of steepest descent $\frac{5}{4}\pi - \theta$ at x_+ and $\theta - \frac{1}{4}\pi$ at x_-. We choose the path of integration:

$$\mathscr{L}_1^{\tan\theta} \bigcup \mathscr{M}_{\tan\theta}^{-\tan\theta} \bigcup \mathscr{M}_{-\tan\theta}^{\tan\theta} \bigcup \mathscr{L}_{\tan\theta}^{1}.$$

Actually, when $x \in \mathscr{L}_1^{\tan\theta}$ or $\mathscr{L}_{\tan\theta}^1$, we see that

$$\mathrm{Re}\{p(x) - p(x_\pm)\} = \ln \frac{x}{|1-x|} - \ln(1 - u + ux) > 0,$$

while on the circle $\{x : |x| = \tan\theta\}$, setting $x = \tan\theta\, e^{i\phi}$ with $\phi \in [0, 2\pi)$,

$$\mathrm{Re}\{p(x) - p(x_\pm)\} = -\tfrac{1}{2}\ln(1 - 4u(1-u)\cos^2\phi)$$

attains its minimum at $\phi = \pm\pi/2$.

Let $\rho = \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}$, notice the polynomial

$${}_1\mathcal{F}_0^{(2/\beta')}\big(-N; \frac{s}{\rho N}; \frac{1-t}{1-u+ut}\big) = {}_0\mathcal{F}_0^{(2/\beta')}\big(\frac{s}{\rho}; \frac{t-1}{1-u+ut}\big) + O(N^{-1}),$$

and

$$B_N \sim (\Gamma_{\beta',n})^{-1} 2^{-\beta' n(\lambda_1+\lambda_2+2)/2 - \beta' n(n-1)/4 - n(2N-3/2)} N^{\beta' n(n-1)/4 + n/2} \exp\{i\pi\big(\beta' n(\lambda_2+1)/2 + n(N-1)\big)\},$$

we are ready to compute the bulk scaling.

For $n = 2m$,

$$\varphi_N\big(u + \frac{s}{\rho N}\big) \sim \Psi_{N,2m}\gamma_m(\beta')\ {}_0\mathcal{F}_0^{(2/\beta')}(i\pi s; 1^m, (-1)^m), \tag{4.42}$$

where

$$\Psi_{N,2m} = (\pi\rho)^{\beta' m(m+1)/2 - m} N^{\beta' m^2/2} 2^{-\beta' m^2/2 - \beta' m(\lambda_1+\lambda_2+1) + 2m(1-2N)},$$

while for $n = 2m-1$

$$\varphi_{N-l}\big(u + \frac{s}{\rho N}\big) \sim \Psi_{N,n}^{(l)} \frac{1}{2\sqrt[4]{u}}\Big(e^{i\theta_N + i(\frac{\pi}{2}-\theta)l} E_{m-1}^{(2/\beta')}(-i\pi s) - e^{-i\theta_N - i(\frac{\pi}{2}-\theta)l} E_{m-1}^{(2/\beta')}(i\pi s)\Big), \tag{4.43}$$

where

$$\begin{aligned}\Psi_{N,2m-1}^{(l)} &= \tbinom{2m-1}{m}\Gamma_{\beta',m-1}\Gamma_{\beta',m}(\Gamma_{\beta',2m-1})^{-1}(\pi\rho)^{\beta'(m-1)^2/2 + (\beta'-2m+1)/2}\sqrt{2}(2\sqrt{u})^{-\beta'/2+1} \\ &\quad \times N^{\beta' m(m-1)/2} i^{-1}(-1)^{n(N-1)} 2^{-\beta' m(m+1)/2 - \beta'(2m-1)(\lambda_1+\lambda_2)/2 + (2m-1)(1-2N+2l)+1}(1-u)^{nl/2}\sqrt[4]{u}.\end{aligned}$$

Here

$$\theta_N = 2(N-1)\theta + (1 + \beta'(m+\lambda_2))(\theta - \pi/4) + \beta'(\lambda_1 - \lambda_2)\theta/2, \qquad \theta = \arccos\sqrt{u}.$$

As previously, when β is even, scaling limits of correlation functions in the bulk for the JβE immediately follow from (4.42). Let $n = 2m = k\beta$, we have

$$R_{k,N}(x_1, \ldots, x_k) = \frac{(k+N)!}{N!}\frac{S_N(\lambda_1,\lambda_2,\beta/2)}{S_{k+N}(\lambda_1,\lambda_2,\beta/2)}\prod_{1\le j<l\le k}(x_j - x_l)^\beta \left[\varphi_N(s_1,\ldots,s_n)\right]_{\{s\}\mapsto\{x\}}. \tag{4.44}$$

Notice

$$\frac{S_N(\lambda_1,\lambda_2,\beta/2)}{S_{k+N}(\lambda_1,\lambda_2,\beta/2)} \sim \pi^{-k}(\Gamma(1+\beta/2))^k 2^{\beta(2N-1+k)k + 2(\lambda_1+\lambda_2+1)k}(\beta N)^{-\beta k/2},$$

in the bulk taking $\rho = \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}$, one obtains

$$\big(\frac{1}{\rho N}\big)^k R_{k,N}\big(u + \frac{x}{\rho N}\big) \sim b_k(\beta)\,|\Delta(2\pi x)|^\beta\, {}_0\mathcal{F}_0^{(\beta/2)}(i\pi s; 1^m, (-1)^m)_{\{s\}\mapsto\{x\}}. \tag{4.45}$$

The use of Corollary 2.3 finally allows us to rewrite the right-hand side in terms of a ${}_1F_1$ hypergeometric function as in Corollary 1.5.

We have thus completed the proof of Theorems 1.1, 1.2, 1.3 and Corollaries 1.4, 1.5 stated in the introduction.

4.4. **The universal multivariate "kernel" in the bulk.** Let us conclude this section by exhibiting a universal pattern observed in the bulk of the classical β-ensembles when $n = 2m-1$ is odd. From the asymptotic results contained in Eqs (4.12), (4.30) and(4.43), one easily establishes the following theorem.

Theorem 4.1 ("Kernel" in the bulk)**.** *Assume that $n = 2m-1$ is odd. Let $A = \sqrt{2N}, 4N, 1$ and $\rho = \frac{2}{\pi}\sqrt{1-u^2}, \frac{2}{\pi}\sqrt{\frac{1-u}{u}}$ and $\frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}$ for the* HβE, LβE, JβE, *respectively. Moreover, let $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$. Then as $N \to \infty$,*

$$\frac{1}{\Psi^{(0)}_{N,2m-1}\Psi^{(1)}_{N,2m-1}}\Big\{\varphi_N\big(A\,u+\frac{A\,s}{\rho N}\big)\varphi_{N-1}\big(A\,u+\frac{A\,t}{\rho N}\big)-\varphi_N\big(A\,u+\frac{A\,t}{\rho N}\big)\varphi_{N-1}\big(A\,u+\frac{A\,s}{\rho N}\big)\Big\}$$
$$\sim\frac{1}{2i}\Big\{E^{(\beta/2)}_{m-1}(i\pi s)\,E^{(\beta/2)}_{m-1}(-i\pi t)-E^{(\beta/2)}_{m-1}(i\pi t)\,E^{(\beta/2)}_{m-1}(-i\pi s)\Big\} \tag{4.46}$$

where $E^{(\alpha)}_k$ denotes the generalized exponential defined in (2.19) *and $\Psi^{(l)}_{N,2m-1}$ is given in* (A.9)*.*

The last theorem obviously generalizes the standard result valid for the unitary (i.e., $\beta = 2$) ensembles and according to which the polynomial kernel[1] $K_N(x,y)$ asymptotically tends to the sine kernel $\frac{\sin\pi(x-y)}{\pi(x-y)}$ when it is rescaled in the bulk of the spectrum (see for instance [23, 41]).

5. PDEs at the edges and in the bulk

Since Kaneko [28] and Yan [51], we know that the hypergeometric function ${}_0F_1^{(\beta/2)}((2/\beta)(\lambda_1+n);s)$ satisfies the following holonomic system of PDEs:

$$s_k\frac{\partial^2 F}{\partial s_k^2}+\frac{2}{\beta}(1+\lambda_1)\frac{\partial F}{\partial s_k}-F+\frac{2}{\beta}\sum_{j=1,j\neq k}^{n}\frac{1}{s_k-s_j}\left(s_k\frac{\partial F}{\partial s_k}-s_j\frac{\partial F}{\partial s_j}\right)=0,\quad k=1,\ldots,n. \tag{5.1}$$

Now, we also know from Theorem 1.3 and Eq. (1.7) that the expectation value φ_N, when rescaled at the hard edge of the LβE or JβE, is given by $(s_1\cdots s_n)^{\lambda_1/\beta}\,{}_0F_1^{(\beta/2)}((2/\beta)(\lambda_1+n);-s)$. Consequently, the limit of the rescaled φ_N satisfies the following system of PDEs:

$$s_k\frac{\partial^2 F}{\partial s_k^2}+\frac{2}{\beta}\frac{\partial F}{\partial s_k}+\left(1-\frac{\lambda_1}{\beta}(\frac{\lambda_1+2}{\beta}-1)\frac{1}{s_k}\right)F+\frac{2}{\beta}\sum_{j\neq k}\frac{1}{s_k-s_j}\left(s_k\frac{\partial F}{\partial s_k}-s_j\frac{\partial F}{\partial s_j}\right)=0,\quad k=1,\ldots,n. \tag{5.2}$$

Similar results hold in the bulk and at the soft-edge of the classical β-ensembles. This was first shown in [36] by exploiting Kaneko's system of PDEs for the hypergeometric function ${}_2F_1$ [28] [2]. More explicitly, by properly rescaling φ_N and then taking the limit $N\to\infty$, one arrives at the conclusion that the limit of the rescaled function φ_N satisfies in the bulk ($n = 2m$),

$$\frac{\partial^2 F}{\partial s_k^2}+F+\frac{2}{\beta}\sum_{j=1,j\neq k}^{n}\frac{1}{s_k-s_j}\left(\frac{\partial F}{\partial s_k}-\frac{\partial F}{\partial s_j}\right)=0,\quad k=1,\ldots,n, \tag{5.3}$$

while at the edge, it satisfies

$$\frac{\partial^2 F}{\partial s_k^2}-s_kF+\frac{2}{\beta}\sum_{j=1,j\neq k}^{n}\frac{1}{s_k-s_j}\left(\frac{\partial F}{\partial s_k}-\frac{\partial F}{\partial s_j}\right)=0,\quad k=1,\ldots,n. \tag{5.4}$$

[1]Following our notation, $K_N(x,y)$ is equal to $\frac{\varphi_N(x)\varphi_{N-1}(y)-\varphi_N(y)\varphi_{N-1}(x)}{x-y}$ with $n=1$.

[2]It was assumed in [36] that the scaling limits of φ_N should exist in the bulk and at the edge. Theorems 1.2 and 1.1 now make this assumption superfluous.

By summing up n equations in (5.4), one easily shows that the multivariate Airy function is one possible solution of the following (single) PDE:

$$D_0 F = p_1(s) F\,. \tag{5.5}$$

This PDE was first given [10, Eq. (5.17)] as an equation satisfied by the Airy function defined in (1.12).

In the one-dimensional case (5.3) and (5.4) respectively reduce to the well-known differential equations satisfied by the sine (or cosine) and Airy functions. However, in the higher-dimensional case, it seems difficult to find all solutions of (5.3) and (5.4), one of which being our limiting expectation value.

Let us consider the limiting bulk expectation value with $n = 2$. In this case, we can express $F(s_1, s_2) = {}_0\mathcal{F}_0^{(\beta/2)}(i, -i; s_1, s_2)$ in a more explicit form. Firstly, according to (2.12), we have

$$F(s_1, s_2) = e^{-is_1 + is_2}\,{}_0\mathcal{F}_0^{(\beta/2)}(2i, 0; s_1 - s_2, 0),$$

which only depends on $s_1 - s_2$. Set $F(s_1, s_2) = f(s_1 - s_2)$, so that $f(x)$ is an analytic function with $f(0) = 1$ and $f(x) = f(-x)$. Secondly, (5.3) implies that $f(x)$ satisfies

$$f'' + \frac{(4/\beta)}{x} f' + f = 0, \tag{5.6}$$

which can be reduced to the Bessel equation (cf. (4.5.9) in [2]). From this, we get

$$f(x) = 2^{\frac{2}{\beta} - \frac{1}{2}} \Gamma(\frac{2}{\beta} + \frac{1}{2})\, x^{\frac{1}{2} - \frac{2}{\beta}} J_{\frac{2}{\beta} - \frac{1}{2}}(x),$$

where $J_\alpha(x)$ is the Bessel function of first kind of order α. Furthermore,

$${}_0\mathcal{F}_0^{(\beta/2)}(i, -i; s_1, s_2) = 2^{\frac{2}{\beta} - \frac{1}{2}} \Gamma(\frac{2}{\beta} + \frac{1}{2})\, (s_1 - s_2)^{\frac{1}{2} - \frac{2}{\beta}} J_{\frac{2}{\beta} - \frac{1}{2}}(s_1 - s_2). \tag{5.7}$$

The right-hand side of (5.7) has been first obtained by Aomoto [3] (at zero) and by Su [47] in the bulk of the HβE with $0 < \beta < 4$.

The case $n = 2$ at the soft edge has also been previously studied. Indeed, Su [47] has proved for the HβE, with the aid of Dumitriu and Edelman's tri-diagonal matrix model, that the scaling limit of $\varphi_N(s)$ has a single integral representation:

$$\frac{1}{4\pi^{3/2} i} \int_{1-i\infty}^{1+i\infty} \frac{e^{\frac{1}{12} z^3 - \frac{1}{2}(s_1 + s_2) - \frac{1}{4z}(s_1 - s_2)^2}}{z^{\frac{2}{\beta} + \frac{1}{2}}} dz.$$

This integral was first defined by Kösters [35]; it is believed to be proportional to our 2-dimensional integral representation.

We end this section with a few remarks on the $\beta = \infty$ case. It is well known that the parameter β of Random Matrix Theory can be interpreted in Statistical Mechanics as the inverse temperature of a log-gas system. Thus $\beta = \infty$ corresponds to the completely frozen state of the system, which is the state where the particles no longer move. Moreover, given that $P_\kappa^{(\infty)}(s) = m_\kappa(s)$, we have

$${}_0\mathcal{F}_0^{(\infty)}(y; s) = \sum_\kappa \frac{1}{\kappa_1! \cdots \kappa_n!} \frac{m_\kappa(y) m_\kappa(s)}{m_\kappa(1^n)}. \tag{5.8}$$

We claim that

$${}_0\mathcal{F}_0^{(\infty)}(y; s) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} e^{s_j y_{\sigma(j)}}. \tag{5.9}$$

Actually, the latter identity follows from the expansion of the right-hand side of (5.9), the use of (5.8) and the following easily established fact

$$\frac{m_\kappa(y)}{m_\kappa(1^n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} y_j^{k_{\sigma(j)}}.$$

By using (5.9), we easily get

$$\mathrm{Ai}^{(\infty)}(s) = \prod_{j=1}^{n} \mathrm{Ai}(s_j), \tag{5.10}$$

where $\mathrm{Ai}(x)$ denotes the one-variable Airy function of first kind ($\mathrm{Bi}(x)$ for the second kind). This result is of course consistent with the physical phenomenon of complete separation of the particles at zero temperature. However, the situation is a little more complicated in the bulk. For instance, when $n = 2$, the use of (5.7) implies that

$${}_0\mathcal{F}_0^{(\infty)}(i, -i; s_1, s_2) = 2^{-\frac{1}{2}}\Gamma(\frac{1}{2})\sqrt{s_1 - s_2}\, J_{-\frac{1}{2}}(s_1 - s_2) = \cos(s_1 - s_2), \tag{5.11}$$

in which the variables cannot separate. We refer to [14] for more information on large β asymptotics.

Another way of putting this is that when $\beta = \infty$, the systems of PDEs such as (5.3) and (5.4) can be directly solved. Indeed, the linearly independent symmetric solutions are respectively

$$\frac{1}{n!}\sum_{\sigma \in S_n} e^{i(s_{\sigma(1)}+\cdots+s_{\sigma(j)}-s_{\sigma(j+1)}-\cdots-s_{\sigma(n)})} = {}_0\mathcal{F}_0^{(\infty)}(1^j, (-1)^{n-j}; is), \quad j = 0, 1, \ldots, n, \tag{5.12}$$

and

$$\frac{1}{n!}\sum_{\sigma \in S_n} \mathrm{Ai}(s_{\sigma(1)})\cdots\mathrm{Ai}(s_{\sigma(j)})\mathrm{Bi}(s_{\sigma(j+1)})\cdots\mathrm{Bi}(s_{\sigma(n)}), \quad j = 0, 1, \ldots, n. \tag{5.13}$$

Note that the equality in (5.12) comes from (5.9).

Now a natural question arises: Can we guess the similar results to (5.12) and (5.13) for general β? In particular, are ${}_0\mathcal{F}_0^{(\beta/2)}((-1)^j, 1^{n-j}; is)$ where $j = 0, 1, \ldots, n$ all linearly independent symmetric solutions of (5.3)? Note that if $j = 0$ or n then

$${}_0\mathcal{F}_0^{(\beta/2)}((-1)^j, 1^{n-j}; is) = e^{\pm i p_1(s)}$$

satisfies (5.3). Moreover, according to our bulk scaling at zero for the HβE and for $n = 2m$ or $2m - 1$

$${}_0\mathcal{F}_0^{(\beta/2)}((-1)^m, 1^{n-m}; \pm is)$$

are also solutions of (5.3), see (4.10) and (4.12) ($\theta = 0$, $l = 0, 1$). So we give the following

Conjecture: The set of symmetric solutions of the system of PDEs (5.3) is spanned by $n+1$ linearly independent functions

$${}_0\mathcal{F}_0^{(\beta/2)}((-1)^j, 1^{n-j}; is), \qquad j = 0, 1, \ldots, n.$$

Acknowledgments. The work of P. D. was supported by FONDECYT grant #1090034 and by CONICYT through the Anillo de Investigación ACT56. The work of D.-Z. L. was supported by FONDECYT grant #3110108 and partially by the National Natural Science Foundation of China (Grant No. 11171005).

Appendix A. Notation and constants

Most of the constants used in the article can be derived from Selberg's integral [23, 41]:

$$\begin{aligned} S_N(\lambda_1, \lambda_2, \lambda_3) :&= \int_{[0,1]^N} \prod_{i=1}^{N} x_i^{\lambda_1}(1 - x_i)^{\lambda_2} \prod_{1 \le i < j \le N} |x_i - x_j|^{2\lambda_3} \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(1 + \lambda_3 + j\lambda_3)\Gamma(1 + \lambda_1 + j\lambda_3)\Gamma(1 + \lambda_2 + j\lambda_3)}{\Gamma(1 + \lambda_3)\Gamma(2 + \lambda_1 + \lambda_2 + (N + j - 1)\lambda_3)}. \end{aligned} \tag{A.1}$$

In articular, one readily shows that

$$W_{\lambda_1,\beta,N} = (2/\beta)^{(1+\lambda_1)N + \beta N(N-1)/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + \beta/2 + j\beta/2)\Gamma(1 + \lambda_1 + j\beta/2)}{\Gamma(1 + \beta/2)}, \tag{A.2}$$

and

$$G_{\beta,N} = \beta^{-N/2-\beta N(N-1)/4}(2\pi)^{N/2}\prod_{j=0}^{N-1}\frac{\Gamma(1+\beta/2+j\beta/2)}{\Gamma(1+\beta/2)}. \tag{A.3}$$

For the Gaussian case, it is often more convenient to use the following integral:

$$\int_{\mathbb{R}^n}\prod_{i=1}^{n} e^{-zx_i^2/2}\prod_{1\le i<j\le n}|x_i-x_j|^\beta\, dx_1\cdots dx_n = \frac{1}{z^{(n+\beta n(n-1)/2)/2}}\Gamma_{\beta,n}, \qquad \mathrm{Re}\{z\}>0, \tag{A.4}$$

where

$$\Gamma_{\beta,n} = (2\pi)^{n/2}\prod_{j=1}^{n}\frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}. \tag{A.5}$$

The Morris normalization constant is

$$M_n(a,b,\alpha) = \prod_{j=0}^{n-1}\frac{\Gamma(1+\alpha+j\alpha)\Gamma(1+a+b+j\alpha)}{\Gamma(1+\alpha)\Gamma(1+a+j\alpha)\Gamma(1+b+j\alpha)}. \tag{A.6}$$

The constants for the the soft edge are

$$\Phi_{N,n} = \begin{cases} N^{\beta' n(n-1)/12+n/6}\exp\{-nN(1+\ln 2-\ln N-2i\pi)/2\} & \text{H}\beta\text{E}\\ 2^{-\beta' n(n-1)/6-\beta' n/2+2n/3}N^{\beta' n(n-1)/12+\beta' n\lambda_1/4+n/6}\exp\{-nN(1-\ln N-i\pi)\} & \text{L}\beta\text{E}\end{cases} \tag{A.7}$$

In the bulk with $n=2m$, we have

$$\Psi_{N,2m} = \begin{cases} (\pi\rho)^{\beta' m(m+1)/2-m}N^{\beta' m^2/2}\exp\{-mN(1+\ln 2-\ln N)\} & \text{H}\beta\text{E}\\ (\pi\rho/2)^{\beta' m(m+1)/2-m}N^{\beta' m(m+\lambda_1)/2}\exp\{-2mN(1-\ln N)\} & \text{L}\beta\text{E}\\ (\pi\rho)^{\beta' m(m+1)/2-m}N^{\beta' m^2/2}2^{-\beta' m^2/2-\beta' m(\lambda_1+\lambda_2+1)+2m(1-2N)} & \text{J}\beta\text{E}\end{cases} \tag{A.8}$$

where $\rho = \frac{2}{\pi}\sqrt{1-u^2}$, $\frac{2}{\pi}\sqrt{\frac{1-u}{u}}$, $\frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}$, respectively correspond to the HβE, LβE and JβE.
The constants for the bulk with $n=2m-1$ are

$$\Psi^{(l)}_{N,2m-1} = \begin{cases} \binom{2m-1}{m}\Gamma_{\beta',m-1}\Gamma_{\beta',m}(\Gamma_{\beta',2m-1})^{-1}(\pi\rho)^{\beta'(m^2-1)/2-(2m-1)/2}N^{\beta' m(m-1)/2} & \\ \times\exp\{-(2m-1)N(1+\ln 2-\ln N)/2\}\,(\sqrt{N/2})^{-(2m-1)l}(2i\sqrt{\pi\rho/2}) & \text{H}\beta\text{E}\\ \binom{2m-1}{m}\Gamma_{\beta',m-1}\Gamma_{\beta',m}(\Gamma_{\beta',2m-1})^{-1}(\pi\rho/2)^{\beta'(m^2-1)/2-m+1}\sqrt{2}(2\sqrt{u})^{-\beta'/2+1} & \\ \times N^{\beta' m(m-1)/2+\beta'(2m-1)\lambda_1/4}\exp\{-(2m-1)N(1-\ln N-i\pi)\}\,(-N)^{-(2m-1)l} & \text{L}\beta\text{E}\\ \binom{2m-1}{m}\Gamma_{\beta',m-1}\Gamma_{\beta',m}(\Gamma_{\beta',2m-1})^{-1}(\pi\rho)^{\beta'(m-1)^2/2+(\beta'-2m+1)/2}\sqrt{2}(2\sqrt{u})^{-\beta'/2+1}N^{\beta' m(m-1)/2} & \\ \times i^{-1}(-1)^{n(N-1)}2^{-\beta' m(m+1)/2-\beta'(2m-1)(\lambda_1+\lambda_2)/2+(2m-1)(1-2N+2l)+1}(1-u)^{nl/2}\sqrt[4]{u} & \text{J}\beta\text{E}\end{cases} \tag{A.9}$$

For the hard edge, the constants are

$$\xi_{N,n} = \begin{cases} \frac{W_{\lambda_1+n,\beta,N}}{W_{\lambda_1,\beta,N}} & \text{L}\beta\text{E}\\ \frac{S_N(\lambda_1+n,\lambda_2;\beta/2)}{S_N(\lambda_1,\lambda_2;\beta/2)} & \text{J}\beta\text{E}.\end{cases} \tag{A.10}$$

Finally, the universal coefficients are

$$a_k(\beta) = (\beta/2)^{(\beta k+1)k}(\Gamma(1+\beta/2))^k\prod_{j=1}^{2k}\frac{(\Gamma(1+2/\beta))^{\beta/2}}{\Gamma(1+\beta j/2)}, \tag{A.11}$$

$$b_k(\beta) = (\beta/2)^{\beta k(k-1)/2}(\Gamma(1+\beta/2))^k\prod_{j=0}^{k-1}\frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta(k+j)/2)}, \tag{A.12}$$

and

$$\gamma_m(\beta') = \binom{2m}{m}\prod_{j=1}^{m}\frac{\Gamma(1+\beta' j/2)}{\Gamma(1+\beta'(m+j)/2)}. \tag{A.13}$$

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Instituto Matemática y Física, Universidad de Talca, 2 Norte 685, Talca, Chile
E-mail address: `Patrick.Desrosiers@inst-mat.utalca.cl`

Instituto Matemática y Física, Universidad de Talca, 2 Norte 685, Talca, Chile
E-mail address: `dzliu@inst-mat.utalca.cl`